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Mathematics

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Abstract

Full Text

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SOME ESTIMATES CONCERNING THE DIRICHLET PROBLEM

1. Consider, in a bounded domain G of variation of n variables x_i , the quasilinear equation

$$\sum a_{ik} u_{ik} = \varphi. \quad (1)$$

It is assumed that the matrix $\|a_{ik}\|$ has no negative eigenvalues (at least, only such solutions u are considered for which this is so). Further, X denotes a point of the domain G , and D a domain contained in G together with its closure.

The solutions $u(X)$ under consideration are assumed to be continuous and to satisfy one of the following conditions:

- (I) u has generalized second derivatives in the sense of S. L. Sobolev, summable with the n -th power in every D .
 - (II) u is twice differentiable.
2. Let L be an m -dimensional plane passing through the origin of coordinates O , and let T be some $(n-1)$ -dimensional plane not passing through O and intersecting L . Rotating L about O so that the intersection LT sweeps out T one-to-one, we obtain an $(n-m)$ -dimensional set of planes L , which we shall call a bundle. In the bundle there is naturally defined an $(n-m)$ -dimensional measure of the set of planes L .

Further, denote by a_L the principal minor of the matrix $\|a_{ik}\|$ corresponding to the indices $1, \dots, m$, if the axes x_1, \dots, x_m are placed in the plane L by a rotation of all the axes.

In all subsequent theorems the following is understood:

If solutions of equation (1) with condition (I) are meant, then the relations appearing in the theorem and depending on L are fulfilled for a set $\{L\}$ having positive measure in some bundle.

If, however, solutions with condition (II) are meant, then it is sufficient to regard such relations as fulfilled for some one plane L .

For each plane L the coordinate axes are rotated so that the axes x_1, \dots, x_m are parallel to L . The dimension m of the planes L is always arbitrary, $1 \leq m \leq n$. For $m = n$, L reduces to the whole space, and the qualifications concerning the set $\{L\}$ and the choice of axes drop out, while $a_L = \text{Det} \|a_{ik}\|$.

If U is the domain of variation of n variables y_1, \dots, y_n , then by

$$\int_U f(y_1, \dots, y_m) dy_1 \cdots dy_m$$

we shall mean the integral over all values (y_1, \dots, y_m) that occur in U .

3. All further results are based on the following lemma. Let $\bar{u}(X)$ be the convex (concave) function "stretched over $u(X)$ from below (from above)," i.e. the greatest (least) of the convex (concave) functions $v(X) \leq u(X)$ ($v \geq u$), $X \in D$. Let $\psi(D, u)$ be its normal image (for the definition see, for example, (1)). Under conditions (I) or (II) imposed on u , $\psi(D, u)$, up to a set of measure zero, is the set of points with coordinates $\bar{u}_i(X)$, $X \in D$.

Lemma. Suppose that for the given solution $u(X)$ of equation (1) the inequality

$$a_L^{-1/m} \varphi \leq P_L(x_1, \dots, x_m) Q_L(u_1, \dots, u_n), \quad (2)$$

where $P_L, Q_L \geq 0$, is satisfied. Then for every D and for almost all L for which (2) holds,

$$\int_D P_L^m dx_1 \dots dx_m \geq m^m \int_{\psi(D, u)} Q_L^{-m}(u_1, \dots, u_m, 0, \dots, 0) du_1 \dots du_m, \quad (3)$$

where $\psi(D, u)$ is taken for the convex \bar{u} stretched over $u(X)$ from below. If, however, $a_L^{-1/m} \varphi \geq -P_L Q_L$, then (3) is true for the concave \bar{u} stretched over $u(X)$ from above.

Inequality (2) is understood to be satisfied up to a set of measure zero, so that it is not excluded, for example, that a_L vanishes somewhere. The integrals in (3) may be infinite.

When the solution $u(X)$ inside the domain departs far from the boundary values, then $\psi(D, u)$ increases. Therefore (3) implicitly contains an estimate for the deviations of $u(X)$ from the boundary values.

4. **Theorem 1.** Suppose that for the given solution $u(X)$ of equation (1), (2) is satisfied, and P_L^m is summable over every D , while $Q(u_1, \dots, u_m, 0, \dots, 0)$ is not summable in the plane u_1, \dots, u_m in any neighborhood of the origin. Then $u(X)$ attains its exact lower bound on the boundary G ; and if $a_L^{-1/m} \varphi \geq -P_L Q_L$ under analogous conditions, then $u(X)$ also attains its upper bound on the boundary.

From Theorem 1 one can derive conditions for uniqueness of the solution of the Dirichlet problem. Thus, for example, the following holds.

Theorem 2. The Dirichlet problem for equation (1) has at most one solution satisfying condition (I), if:

- 1) $a = \text{Det} \|a_{ik}\| > \text{const} > 0$ (which may be regarded as satisfied, if $a > 0$, after merely dividing (1) by $a^{1/n}$);
- 2) a_{ik} do not depend on u , and φ is nondecreasing in u ;
- 3) in every domain D , for bounded u, u_j ,

$$|a_{ik}(u_j + \Delta u_j, x_j) - a_{ik}(u_j, x_j)| \leq M \left[\sum \Delta u_j^2 \right]^{1/2},$$

$$|\varphi(u_j + \Delta u_j, u, x_j) - \varphi(u_j, u, x_j)| \leq N(x_j) \left[\sum \Delta u_j^2 \right]^{1/2},$$

where M is a constant, and $N(x_j)$ is summable with the n -th power (M and N , generally speaking, depend on D and on the bounds for u, u_j).

5. **Theorem 3.** Suppose that for certain solutions $u(X)$ of equation (1) inequality (2) is satisfied with the same functions P_L, Q_L for all of them, and

$$\int_{G_j} P_L^m dx_1 \dots dx_m < m^m \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} Q_L^{-m}(u_1, \dots, u_m, 0, \dots, 0) du_1 \dots du_m,$$

which is certainly true if the left-hand integral is finite and the right-hand one is infinite. Then for all such solutions the quantity $\inf_G u(X) - \inf_{\Gamma} u(X)$ is bounded below by one and the same number. Analogously, $\sup_G u(X) - \sup_{\Gamma} u(X)$ is bounded above if $a_L^{-1/m} \varphi \geq -P_L Q_L$ under the same conditions on P_L, Q_L .

6. Let us consider, in particular, the linear equation

$$L(u) \equiv \sum a_{ik} u_{ik} + \sum b_i u_i + cu = f. \quad (4)$$

Introduce the notation: $[\sum b_i]^{1/2} = b$, and for any function $g(x_1, \dots, x_n)$, under the condition that the axes x_1, \dots, x_m lie in the plane L , set

$$g_L(x_1, \dots, x_m) = \sup_{(x_{m+1}, \dots, x_n)} g(x_1, \dots, x_n).$$

From Theorem 1 one easily obtains:

Theorem 4. *Suppose that in equation (4) $c \leq 0$ and $a_L^{-1}b_L^m$ is summable over each domain D . Then, for $f = 0$, no solution can attain inside the domain a negative (positive) minimum (maximum) without attaining it on the boundary, and the Dirichlet problem therefore cannot have more than one solution.*

For $m = n$, Theorem 4 reduces to the assertion that, for $c \leq 0$, uniqueness of the solution of the Dirichlet problem is ensured by the summability of $a^{-1/n}b$ to the n -th power, where $a = \text{Det } \|a_{ik}\|$. At the same time, simple examples show that this requirement can no longer be replaced by summability to any power smaller than n . Moreover, as is shown by an example pointed out to me by Yu. G. Reshetnyak, the summability of the generalized derivatives u_{ik} to the n -th power assumed by us also cannot be replaced by summability to a smaller power. Finally, only a_L enters the theorem (for $m = n$, respectively, $\text{Det } \|a_{ik}\|$), which is essential when the boundedness of the coefficients a_{ik} is abandoned. Thus, Theorem 4 gives, in a certain sense, minimal conditions for uniqueness of the solution of the Dirichlet problem for $c \leq 0$.

7. Introduce the notation: $c_+ = c$ for $c > 0$, and $c_+ = 0$ for $c \leq 0$,

$$B_L = \int_G \frac{b_L^m}{a_L} dx_1 \dots dx_m, \quad C_L = \int_G \frac{c_+^m}{a_L} dx_1 \dots dx_m, \quad F_L = \int_G \frac{|f|_L^m}{a_L} dx_1 \dots dx_m.$$

Let H_L also denote the convex hull of the projection of the domain G onto the plane L .

Theorem 5. *There exists a decreasing positive function $\Phi(B_L; H_L)$, such that uniqueness of the solution of the Dirichlet problem for equation (4) is ensured by the condition $C_L < \Phi(B_L; H_L)$.*

Since $\Phi > 0$, for $C_L = 0$ this condition is satisfied automatically, provided that $B^L < \infty$. (This last remark ensures uniqueness of the solution of the Dirichlet problem for $c \leq 0$, if $a_L^{-1}b_L^m$ is summable over the whole domain G , which, however, is stronger than the condition of Theorem 4.)

One can give an explicit expression for the function Φ , but it is rather complicated. For $B_L = 0$, i.e. $b = 0$, the condition $C_L < \Phi$ can be put in the simple form:

$$C_L \leq m^m \kappa_m^2 V_L^{-1}, \quad (5)$$

where κ_m is the volume of the m -dimensional unit ball, and V_L is the volume of the m -dimensional ellipsoid containing H_L .

Theorem 5 in an obvious way includes a lower estimate for the first eigenvalue of the equation $L(u) + \lambda u = 0$.

Theorems similar to Theorem 5 are well known for elliptic equations under more restrictive assumptions on the coefficients and on the character of the solution. Estimates depending on the volume of the domain are known (see, for example, (2, 3)). The estimate contained in Theorem 5 depends on the convex hull of the

domain, and not on the volume of the domain itself; but for convex domains the nature of the estimate is the same as in the known cases just mentioned.

* That is, in particular, if $H'_L \supset H''_L$, then $\Phi(B_L, H'_L) \leq \Phi(B_L, H''_L)$.

8. **Theorem 6.** If, for a solution $u(X)$ of equation (4), one sets

$$h = \inf_{\Gamma} u(X) - \inf_G u(X), \quad h_{\Gamma} = \inf_{\Gamma} u(X),$$

then

$$h \leq \Psi(B_L, (1 + h_{\Gamma}^n)C_L, F_L; H_L),$$

where Ψ is an increasing function of all its arguments.

The same estimate is valid for $h = \sup_G u(X) - \sup_{\Gamma} u(X)$ when $h_{\Gamma} = \sup_{\Gamma} u(X)$.

In the simplest case, when $b = c_+ = 0$, the estimate can be represented in the form

$$h^m \leq m^{-m} \nu_m^{-2} V_L F_L,$$

where ν_m and V_L are the same as in (5).

9. Let us also note the following result.

Theorem 7. If equation (4) with $b = f = 0$ has a nontrivial, sign-constant solution $u(X)$ with boundary condition $u|_{\Gamma} = 0$, then, putting $\sup |u| = h$, we have

$$\int_G \left(\frac{|u|_L}{h} \right)^{2m} dx_1 \cdots dx_m > m^{m/2} \nu_m V_L^{-1/2} \left[\int_G \frac{|c|_L^{2m}}{a_L^2} dx_1 \cdots dx_m \right]^{-1}. \quad (6)$$

This means that $u(X)$ cannot have too pronounced a maximum (minimum). A similar assertion is valid when $b \neq 0$, but then the estimate for the left-hand side of (6) is more complicated and also includes B_L .

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Note: Figure translations are in progress. See original paper for figures.

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