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Abstract

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ON A CERTAIN FREE-BOUNDARY PROBLEM FOR THE HEAT EQUATION

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Consider the following boundary-value problem: to find functions $u(x,t)$ and $s(t)$ such that the function $s(t)$ is defined and continuous for $t \geq 0$; $s(0) = 0$; $s(t) > 0$ for $t > 0$; the function $u(x,t)$ in the domain $D\{0 < x < s(t),\ 0 < t < T\}$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{1}$$

is continuous together with $\partial u/\partial x$ in the closure $\overline{D}$ of the domain D, and satisfies the conditions

$$u\Big|_{x=0} = f_1(t), \qquad u\Big|_{x=s(t)} = f_2(t), \qquad \frac{\partial u}{\partial x}\bigg|_{x=s(t)} = g(t). \tag{2}$$

Similar problems arise, for example, in the study of filtration with allowance for the influence of bound water (1).

For problem (1), (2) the following existence and uniqueness theorems are valid.

Existence theorem. Let the functions $f_1,\ f_2,\ g \in C^2$ and let the conditions

$$f_1 < 0, \qquad f_2 \leq 0, \qquad g > 0, \qquad f_1(0) = f_2(0) = 0, \qquad f_2 - f_1 > 0 \quad \text{for } t > 0,$$

$$f_1' \leq 0, \qquad f_2' \leq 0; \tag{3}$$

$$f_2'' \geq 0, \qquad (f_2 - f_1)' \geq 0, \qquad g' < 0. \tag{4}$$

be satisfied. Then there exists a solution of problem (1), (2)—a pair of functions $u(x,t)$, $s(t)$. The function $u(x,t)$ is continuous in $\overline{D}$ together with the derivatives entering equation (1); the function $s(t)$ is differentiable, $s(0) = 0$, $s(t) > 0$ for $t > 0$, $ds/dt \geq 0$.

Uniqueness theorem. The solution $u(x,t)$, $s(t)$ of problem (1), (2) is unique if the following conditions are satisfied: the function $u(x,t)$ is continuous in $\overline{D}$ together with the derivatives entering equation (1); the function $s(t)$ is differentiable and satisfies the conditions $s(0) = 0$, $s(t) > 0$ for $t > 0$, $ds/dt \geq 0$; the functions $f_1,\ f_2,\ f_1',\ f_2',\ g'$ are continuous and satisfy the relations (3).

Let us note that, by making the substitution $u_1 = au + b$, $a, b = \text{const}$, and using the theorems formulated above, one can obtain existence and uniqueness theorems also for other classes of boundary conditions.

We shall indicate the ideas of the proof of the formulated theorems.

The existence theorem is proved by means of the method of lines. Equation (1) is replaced by the equation

$$\frac{d^2u(x,n\Delta t)}{dx^2} = \frac{u(x,n\Delta t) - u(x,(n-1)\Delta t)}{\Delta t}, \tag{5}$$

and the boundary conditions (3) by the conditions

$$u(0,n\Delta t) = f_1(n\Delta t), \qquad u(s(n\Delta t),n\Delta t) = f_2(n\Delta t),$$

$$\frac{du(s(n\Delta t),n\Delta t)}{dx} = g(n\Delta t). \tag{6}$$

In what follows we shall use the notation $u(x,n\Delta t) = u_n$; in general, the value of a function at an argument equal to $n\Delta t$ will be denoted by the corresponding letter with subscript n.

Let the function u_{n-1} be known on the interval $0 \leq x \leq s_{n-1}$; in order that u_n on the interval $0 \leq x \leq s_n$ can be determined from equation (5), the function u_{n-1} is extrapolated as follows: u_{n-1} is continued by a parabola so that it is continuous together with its first and second derivatives on the interval from s_{n-1} to the vertex of this parabola. It can be shown that the vertex of the parabola lies to the right of the point s_{n-1}. Further, the function u_{n-1} is continued continuously by a constant. We note that u_{n-1} has the form of a parabola on the interval

$$s_{n-1} \leq x \leq s_{n-1} - \frac{g_{n-1}}{q_{n-1}s_{n-1}},$$

where

$$q_n(x) = \frac{u_n(x) - u_{n-1}(x)}{\Delta t}, \quad n = 1, 2, \dots; \qquad q_0(0) = f_1'(0).$$

Next the existence of a solution of problem (5), (6) is proved. The general solution of equation (5) has the form

$$u_n = C_1 \operatorname{sh} \frac{x}{\sqrt{\Delta t}} + C_2 \operatorname{ch} \frac{x}{\sqrt{\Delta t}} - \frac{1}{\sqrt{\Delta t}} \int_0^x \operatorname{sh} \frac{x-\xi}{\sqrt{\Delta t}}\, u_{n-1}(\xi)\, d\xi.$$

By virtue of the conditions (6) we have

$$f_{1n} = C_2,$$

$$f_{2n} = C_1 \operatorname{sh} \frac{s_n}{\sqrt{\Delta t}} + C_2 \operatorname{ch} \frac{s_n}{\sqrt{\Delta t}} - \frac{1}{\sqrt{\Delta t}} \int_0^{s_n} \operatorname{sh} \frac{s_n-\xi}{\sqrt{\Delta t}}\, u_{n-1}(\xi)\, d\xi, \qquad (7)$$

$$g_n = \frac{C_1}{\sqrt{\Delta t}} \operatorname{ch} \frac{s_n}{\sqrt{\Delta t}} + \frac{C_2}{\sqrt{\Delta t}} \operatorname{sh} \frac{s_n}{\sqrt{\Delta t}} - \frac{1}{\Delta t} \int_0^{s_n} \operatorname{ch} \frac{s_n-\xi}{\sqrt{\Delta t}}\, u_{n-2}(\xi)\, d\xi.$$

Put

$$\Phi_n(s) = f_{2n} \operatorname{ch} \frac{s}{\sqrt{\Delta t}} - \sqrt{\Delta t}\, g_n \operatorname{sh} \frac{s}{\sqrt{\Delta t}} - f_{1n} - \frac{1}{\sqrt{\Delta t}} \int_0^s \operatorname{sh} \frac{\xi}{\sqrt{\Delta t}}\, u_{n-1}(\xi)\, d\xi.$$

From the last two equations (7) it follows that the required point s satisfies the equation $\Phi_n(s_n) = 0$.

By virtue of the method of extrapolation of the function u_{n-1} and of the properties of the functions f_1, f_2, g, for $s \geq s_{n-1}$ we have $\Phi'(s) < 0$ and $\Phi''(s) \leq 0$. If $\Phi_n(s_{n-1}) \geq 0$, then there exists a unique solution s_n of the equation $\Phi_n(s_n) = 0$ such that $s_n \geq s_{n-1}$.

We have

$$\Phi_n(s_{n-1}) = [(f_{2n} - f_{1n}) - (f_{2(n-1)} - f_{1(n-1)})] - \sqrt{\Delta t} \operatorname{sh} \frac{s_{n-1}}{\sqrt{\Delta t}} (g_n - g_{n-1}) +$$

$$+ \frac{1}{\sqrt{\Delta t}} \int_0^{s_{n-1}} \operatorname{sh} \frac{\xi}{\sqrt{\Delta t}} [(f_{2n} - f_{2(n-1)}) - (u_{n-1}(\xi) - u_{n-2}(\xi))]\, d\xi.$$

The first two terms are nonnegative. Moreover, under conditions (3), (4) the function

$$v_n = f_{2n} - f_{2(n-1)} - (u_{n-1} - u_{n-2}) \geq 0;$$

it satisfies the equation

$$\frac{d^2 v_n}{dx^2} - \frac{v_n - v_{n-1}}{\Delta t} = -\Delta t \, \frac{\Delta^2 f_{2n}}{\Delta t^2}.$$

and nonpositive boundary conditions.

The solutions $u_n(x)$ of equation (5) have the following properties:

1) The function $u_n \leqslant 0$.

2) The functions

$$q_n = \frac{u_n - u_{n-1}}{\Delta t} \leqslant 0.$$

Assertions 1) and 2) follow from the maximum principle: the functions satisfy equation (5) and nonpositive boundary conditions.

3) The functions q_n are bounded.

This is proved by generalizing to the solutions of problem (5), (6) the following a priori estimate for the solutions of problem (1), (2). One can show that the function $q = du/dt$ satisfies the equation $\partial^2 q/\partial x^2 = \partial q/\partial t$ and the boundary conditions $q|_{x=0} = f_1'(t)$, $q|_{x=s(t)} = f_2'(t) - g(t)\, ds/dt$. Moreover, differentiating the identity $du(s(t), t)/dx = g(t)$, one can obtain $\partial q/\partial x|_{x=s(t)} = g'(t) - q|_{x=s(t)}\, ds/dt = g'(t) - f_2'(t)\, ds/dt + g(t)(ds/dt)^2$. From this equality it is clear that if q assumes its minimum value at $x = s(t)$, then the quantity ds/dt at the point of minimum of q is bounded, and therefore the function q itself is bounded. From the boundedness of q_n follows the boundedness of $\Delta s_n/\Delta t$.

4) The functions

$$r_n = \frac{q_n - q_{n-1}}{\Delta t}$$

are bounded.

The boundedness of r_n is proved in approximately the same way as the boundedness of q_n. From the boundedness of r_n follows the boundedness of $\Delta^2 s_n/\Delta t$.

From the indicated properties of the solutions of problem (5), (6) there follows the existence of a solution of problem (1), (2), as well as the continuity in $\overline{D}$ of the derivatives of the function $u(x, t)$ entering into equation (1), and the nonpositivity of ds/dt.

The uniqueness theorem is proved as follows. Let $u(x, t), s(t)$ be a solution of problem (1), (2) satisfying the conditions stated in the formulation of the uniqueness theorem. It has the following properties: 1) $u \leqslant 0$; 2) $q = \partial u/\partial t = \partial^2 u/\partial x^2 \leqslant 0$. This follows from the fact that the functions u and q satisfy the heat equation and nonpositive boundary conditions. From properties 1) and 2) it follows that $\partial u/\partial x > 0$. Now let $u_1(x, t), s_1(t)$ and $u_2(x, t), s_2(t)$ be two such solutions of problem (1), (2). Put $s(t) = \min(s_1(t), s_2(t))$; in the domain

$\{0 \leqslant x \leqslant s(t),\ 0 \leqslant t \leqslant T\}$ both functions u_1 and u_2, and their difference $v = u_1 - u_2$, are defined.

Denote by l_1 the set of points of the curve $x = s(t)$ where $s = s_1$, and by l_2 the set of points where $s = s_2$.

On l_1 we have

$$v = u_1 - u_2 = f_2 - u_2 \geqslant 0,$$

$$\frac{\partial v}{\partial x} = \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} = g - \frac{\partial u_2}{\partial x} \leqslant 0, \tag{8}$$

and on l_2

$$v = u_1 - u_2 = u_1 - f_2 \leqslant 0,$$

$$\frac{\partial v}{\partial x} = \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} = \frac{\partial u_1}{\partial x} - q \geqslant 0, \tag{9}$$

where the equalities $f_2 = u_1$ and $f_2 = u_2$ hold only on $l_1 \cap l_2$.

It follows from inequalities (8) and (9) that on the curve $x = s(t)$, $|v|$ cannot have a maximum, since for $x = 0$, $v = 0$; we have $v = 0$ in $\overline{D}$ and $s_1 \equiv s_2$.

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Moscow State University
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Note: Figure translations are in progress. See original paper for figures.

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