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Abstract

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ON THE INDUCTIVE DIMENSION OF PROXIMITY SPACES

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1. Yu. M. Smirnov introduced the notion of the δ-dimension δdP of a proximity space P, defined by means of coverings. He also raised the question of the possibility of defining the inductive dimension $\delta \operatorname{Ind} P$ ([5]), p. 289). In this note I propose a natural definition of inductive dimension such that $\delta \operatorname{Ind} P \geqslant \delta dP$ for every space P. The author knows of no example for which one would have $\delta \operatorname{Ind} P > \delta dP$; equality holds for every subspace of Euclidean space and for every metric space M of finite dimension ΔdM (in the sense of ([4])).

The inequality of N. Vedenisov ([1]), $\operatorname{Ind} X \geqslant \dim X$, proved by him for normal spaces X, follows easily from the inequality $\delta \operatorname{Ind} P \geqslant \delta dP$.

2. Let P be a proximity space. A set C, by definition, δ-**separates** the sets A, B if there exists a decomposition $P \setminus C = A' \cup B'$, A' is far from B', $A' \supset A$, and $B' \supset B$. A set H **frees** A, B if H is far from $A \cup B$ and every δ-neighborhood of the set H which does not meet the set $A \cup B$, δ-separates A and B.

Lemma 1. *If H frees A and B, then the closure of the set H frees the closures of A and B, and conversely.*

A **chain** between A and B is such a finite sequence of sets $\Gamma_1, \dots, \Gamma_n$ that $A \cap \Gamma_1 \neq \Lambda$, $B \cap \Gamma_n \neq \Lambda$, and $\Gamma_i \cap \Gamma_{i+1} \neq \Lambda$ for $i = 1, \dots, n-1$.

Remark. In order that the set H, assumed far from $A \cup B$, free A and B, it is necessary and sufficient that into every δ-covering γ one can inscribe such a δ-covering γ' that every chain of elements of γ' between A and B contains an element intersecting H.

Let the equality $\delta \operatorname{Ind} P = -1$ mean that P is the empty space. We shall say that $\delta \operatorname{Ind} P \leqslant n$ if for every pair of far sets A, B there exists a set H which frees A and B and $\delta \operatorname{Ind} H \leqslant n - 1$. If there is no such number n, then $\delta \operatorname{Ind} P = \infty$.

Lemma 2. *Let the space P be normal, and let the proximity be the largest of those that generate the topology of P. If H separates A and B in the topological sense, then H frees A and B.*

The converse assertion is false. This is easily seen from the example of a sequence of intervals converging to an interval.

By easy induction from the lemma we obtain the following theorem.

Theorem 1. *Let the space P be normal, and let the proximity be the largest of those that generate the topology of P. Then* $\operatorname{Ind} P \geqslant \delta \operatorname{Ind} P$.

Let now uP be the bicompact δ-extension of the δ-space P. It is known ([5]) that $\delta duP = \delta dP$.

Theorem 2. *If the space P is dense in the space $\widetilde{P}$, then $\delta \operatorname{Ind} P \geqslant \delta \operatorname{Ind} \widetilde{P}$; in particular, $\delta \operatorname{Ind} P \geqslant \delta \operatorname{Ind} uP$. Moreover, $\delta \operatorname{Ind} uP \geqslant \delta dP$.*

The first inequality follows almost directly from Lemma 1. To prove the last inequality one must apply induction and the sum theorem ([5]) for δ-dimension. For every δ-covering γ there exists a set H such that H consists of a finite sum of sets H_i having $\delta \operatorname{Ind} H_i \leqslant n-1$, and $P \setminus H$ is a sum of pairwise disjoint sets, lying singly in some $\Gamma_i \in \gamma$.

Corollary (Vedenisov ([1])). *For every normal space X,*

$$\operatorname{Ind} X \geqslant \dim X.$$

3. We shall consider a metric space R as a uniform space. It is known ([2]) that a δ-homeomorphism of metric spaces is simply a uniform homeomorphism. A subset M of a uniform space R is called **uniformly discrete** if every covering of the set M is a uniform covering; a system γ of sets Γ_α is called **uniformly discrete** if the sets Γ_α are pairwise disjoint and γ is a uniform covering of the sum $\bigcup \Gamma_\alpha$. The **large dimension** ΔdR is defined as the least of all such integers $n \geqslant 0$ that in every uniform covering of the space R one can inscribe a uniform covering of multiplicity $\leqslant n+1$.

Lemma 3. *Let R be a uniform space, H a nonempty subset of the space R, and Q such a subset of the space R that every set M lying in Q and far from H is uniformly discrete. Then*

$$\delta \operatorname{Ind} Q \leqslant \delta \operatorname{Ind} H.$$

Lemma 4. *Let R be a metric space, $\widetilde{R}$ the completion of the space R, and H any subset of the completion $\widetilde{R}$. There exists such a set Q of the space R that its closure contains H, and that every set M from Q which is far from H is uniformly discrete.*

Indeed, if H separates two sets and they are far from Q, then Q separates them. By applying induction from Theorem 2 one obtains:

Theorem 3. *Let R be a metric space, and $\widetilde{R}$ its completion. Then*

$$\delta \operatorname{Ind} R = \delta \operatorname{Ind} \widetilde{R}.$$

For the next theorem we need two lemmas. If $\Delta dR \leqslant n$, then in every uniform covering of the space R one can inscribe a uniform covering γ, consisting of the sum of $n+1$ uniformly discrete systems $\gamma_0, \dots, \gamma_n$ ([4]). Secondly, in the proof we shall use the notion of a stable centered system. In 1954 S. Ginsburg and the author studied such systems (unpublished). A centered system φ of sets in a uniform space is called **stable** if for every uniform covering γ there exists a set $A \in \varphi$ such that, for every set $B \in \varphi$, the sum of all elements of the covering γ that meet B contains A.

Lemma 5. *Let R be a complete metric space, φ a stable centered system in the space R, and H the set of all points of proximity to every element of the system φ. Then every δ-neighborhood of the set H contains an element of φ.*

Theorem 4. *For every metric space R we have:*

$$\delta \operatorname{Ind} R = \Delta dR.$$

Sketch of proof. Recalling Theorem 3 and the corresponding theorem from ([3]), we may assume that R is complete. It suffices to prove that if $\Delta dR = n$, then for any two far sets A, B there exists a set H separating them such that $\Delta dH \leqslant n-1$. Then there exists a sequence of coverings γ^i, where γ^{i+1} is inscribed in γ^i and each γ^i consists of the $(n+1)$-st uniformly discrete system $\gamma_0^i, \dots, \gamma_n^i$ ([4]). In order to construct a stable centered system, we remove, first, the δ-neighborhoods of the sets A and B, and then such a subset of the sum of the system γ_0^i that $\gamma_1^i \cup \dots \cup \gamma_n^i$ is a uniform covering of the remaining set D_i, which δ-separates A and B.

Then the desired set H is the set of all points close to every element of this system.

We note that, by the method described above, releasing points of a line, one can obtain a perfect Cantor set!

It is known ([4]) that for any uniform space R of finite dimension ΔdR the equality $\delta dR = \Delta dR$ follows.

Corollary. If the space R is metric and $\Delta dR < \infty$, then
$\delta dR = \Delta dR = \delta \operatorname{Ind} R$.

4. Apart from the results set forth above, little is known about the dimensions Δd and $\delta \operatorname{Ind}$. In ([4]) an example is given of a space R for which $\Delta dR = \infty$, while $\delta dR = 0$. As Yu. M. Smirnov noted ([5]), $\delta dR = 0$ if and only if every two distant sets are δ-separable by the empty set, i.e. when $\delta \operatorname{Ind} R = 0$.

It is not difficult to prove:

Theorem 5. For a metric space R, from the equality $sdR = 0$ it follows that $\Delta dR = 0$.

It is clear from this that if, for a metric space R, the dimensions δdR, ΔdR, $\delta \operatorname{Ind} R$ do not coincide, then

$\delta \operatorname{Ind} R \geq \delta dR \geq 1$, $\Delta dR = \infty$.

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Note: Figure translations are in progress. See original paper for figures.