MATHEMATICS

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Abstract

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ON THE APPLICATION OF THE METHOD OF VARIATION OF A PARAMETER TO THE COMPUTATION OF DETERMINANTS

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1°. Let there be given a square matrix $A(\lambda) = \|a_{ij}(\lambda)\|$ $(i, j = 1, 2, \dots, n)$ of order n, whose elements are functions of a parameter λ, taking prescribed values on some finite interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$. We shall call the matrix $A^{-1}(\lambda) = \|\alpha_{ij}(\lambda)\|$ $(i, j = 1, 2, \dots, n)$ the inverse of the matrix $A(\lambda)$, if

$$A^{-1}(\lambda)A(\lambda) = E, \tag{1}$$

where E is the identity matrix.

Suppose that the functions $a_{ij}(\lambda)$ $(i, j = 1, 2, \dots, n)$ are defined and continuous on the whole interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$, and have continuous derivatives on this interval. Suppose, moreover, that the matrix $A(\lambda)$ has on the interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$ a determinant $\Delta(\lambda)$ different from zero, and that for some value of λ, for example $\lambda = \lambda_0$, the value of this determinant is known to us:

$$\Delta(\lambda_0) = \Delta^{(0)}. \tag{2}$$

It is required to find approximate values of $\Delta(\lambda)$ for prescribed values $\lambda > \lambda_0$.

Denote by $dA(\lambda)/d\lambda = \|a'_{ij}(\lambda)\|$ the matrix obtained from the matrix $A(\lambda)$ by replacing all its elements by their derivatives with respect to λ, and prove the following lemma.

Lemma. If for the matrix $A(\lambda) = \|a_{ij}(\lambda)\|$ $(i, j = 1, 2, \dots, n)$ there exists on the interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$ the inverse matrix $A^{-1}(\lambda) = \|\alpha_{ij}(\lambda)\|$ $(i, j, \dots, n)$, then for all λ from this interval the relation holds*

$$\frac{d\Delta(\lambda)}{d\lambda} = \Delta(\lambda)\,\mathrm{Sp}\left(A^{-1}(\lambda)\frac{dA(\lambda)}{d\lambda}\right)^{**}. \tag{3}$$

Proof. Taking λ as the independent variable and differentiating with respect to λ the determinant

$$\Delta(\lambda) = \begin{vmatrix} a_{11}(\lambda) & a_{12}(\lambda) & \dots & a_{1n}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & \dots & a_{2n}(\lambda) \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}(\lambda) & a_{n2}(\lambda) & \dots & a_{nn}(\lambda) \end{vmatrix},$$

* An analogous lemma was proved in a somewhat different way in (1).
** Here $\operatorname{Sp} D$ denotes the trace of the matrix D, i.e. the sum of its diagonal elements.

we find

$$\frac{d\Delta(\lambda)}{d\lambda} = \sum_{l=1}^{n} \begin{vmatrix} a_{11}(\lambda) & \dots & a_{1,l-1}(\lambda) & a'_{1l}(\lambda) & a_{1,l+1}(\lambda) & \dots & a_{1n}(\lambda) \\ a_{21}(\lambda) & \dots & a_{2,l-1}(\lambda) & a'_{2l}(\lambda) & a_{2,l+1}(\lambda) & \dots & a_{2n}(\lambda) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}(\lambda) & \dots & a_{n,l-1}(\lambda) & a'_{nl}(\lambda) & a_{n,l+1}(\lambda) & \dots & a_{nn}(\lambda) \end{vmatrix}. \quad (4)$$

Further, differentiating identity (1) with respect to λ, we have

$$\frac{dA^{-1}(\lambda)}{d\lambda} A(\lambda) + A^{-1}(\lambda) \frac{dA(\lambda)}{d\lambda} = 0, \quad (5)$$

whence

$$\frac{dA(\lambda)}{d\lambda} = -A(\lambda) \frac{dA^{-1}(\lambda)}{d\lambda} A(\lambda),$$

or, in expanded form,

$$a'_{ij}(\lambda) = -\sum_{\nu,\mu=1}^{n} a_{i\nu}(\lambda) \alpha'_{\nu\mu}(\lambda) a_{\mu j}(\lambda) \quad (i,\ j = 1,\ 2, \dots, n). \quad (6)$$

Substituting (6) into (4), we obtain

$$\frac{d\Delta(\lambda)}{d\lambda} = -\Delta(\lambda) \sum_{p,q=1}^{n} \alpha'_{pq}(\lambda) a_{qp}(\lambda), \quad (7)$$

where, evidently,

$$\sum_{p,q=1}^{n} \alpha'_{pq}(\lambda) a_{qp}(\lambda) = \operatorname{Sp} \left(\frac{dA^{-1}(\lambda)}{d\lambda} A(\lambda) \right).$$

Taking identity (5) into account, we arrive at the required relation (3). The lemma is proved.

To determine, for given λ, the values of the determinant $\Delta(\lambda)$, we numerically integrate the differential equation (3) on the interval $\lambda_0 \leq \lambda \leq \lambda^*$ under the initial condition (2):

$$\lambda = \lambda_0, \qquad \Delta(\lambda) = \Delta^0.$$

The numerical values $\Delta(\lambda)$ obtained in the integration for each specified value of the parameter λ will be the desired approximate values of the determinant of the matrix $A(\lambda)$.

We note that, in the numerical integration of equation (3), it is necessary to know at each step the elements of the inverse matrix $A^{-1}(\lambda)$. In computing these elements one may use, for example, the method of parameter variation ([2]). In doing so one should additionally assume that for $\lambda = \lambda_0$ the inverse matrix $A^{-1}(\lambda)$ is known:

$$A^{-1}(\lambda_0) = A_0^{-1} = \|\alpha_{ij}^{(0)}\| \quad (i,\ j = 1,\ 2, \dots, n), \tag{8}$$

and, instead of one equation (2), numerically integrate the system of two equations

$$\begin{aligned} \frac{d\Delta(\lambda)}{d\lambda} &= \Delta(\lambda)\,\mathrm{Sp}\left(A^{-1}(\lambda)\frac{dA(\lambda)}{d\lambda}\right), \\ \frac{dA^{-1}(\lambda)}{d\lambda} &= -A^{-1}(\lambda)\frac{dA(\lambda)}{d\lambda}A^{-1}(\lambda) \end{aligned} \tag{9}$$

under the initial conditions (2), (8):

$$\lambda = \lambda_0, \qquad \Delta(\lambda) = \Delta^{(0)}, \qquad A^{-1}(\lambda) = A_0^{-1}.$$

2°. Let now, for some value $\lambda = \lambda_i$ from the interval

for $\lambda_0 \ll \lambda \ll \lambda^*$ the determinant $\Delta(\lambda)$ vanishes, i.e.,

$$\Delta(\lambda_i) = 0.$$

Let us write the matrix $A(\lambda)$ in the form

$$A(\lambda) = \begin{Vmatrix} P(\lambda) & u(\lambda) \\ v(\lambda) & a_{nn}(\lambda) \end{Vmatrix},$$

where $P(\lambda)$ is a matrix of order $(n-1)$, $v(\lambda)$ is a row, $u(\lambda)$ is a column, and suppose that the determinant $\overline{\Delta}(\lambda)$ of the matrix $P(\lambda)$ in some neighborhood of the point λ_i, say $\lambda_i' \ll \lambda \ll \lambda_i''$, is nonzero.

We write the inverse matrix $A^{-1}(\lambda)$, according to the bordering method [3], in the form

$$A^{-1}(\lambda) = \left\| \begin{matrix} P^{-1}(\lambda) + \dfrac{P^{-1}(\lambda)u(\lambda)v(\lambda)P^{-1}(\lambda)}{\alpha(\lambda)} & -\dfrac{P^{-1}(\lambda)u(\lambda)}{\alpha(\lambda)} \\ -\dfrac{v(\lambda)P^{-1}(\lambda)}{\alpha(\lambda)} & \dfrac{1}{\alpha(\lambda)} \end{matrix} \right\|, \qquad (10)$$

where $P^{-1}(\lambda)$ is the inverse matrix to the matrix $P(\lambda)$,

$$\alpha(\lambda) = a_{nn}(\lambda) - v(\lambda)P^{-1}(\lambda)u(\lambda).$$

The relation [4,5] holds

$$\alpha(\lambda) = \frac{\Delta(\lambda)}{\overline{\Delta}(\lambda)}, \qquad (11)$$

from which it is seen that $\Delta(\lambda)$ and $\alpha(\lambda)$ vanish simultaneously.

Thus, in order to find the numerical values of $\Delta(\lambda)$ in a neighborhood of the point λ_i, on the interval $\lambda_i' \ll \lambda \ll \lambda_i''$, instead of the system (9) we numerically integrate the system

$$\frac{d\Delta(\lambda)}{d\lambda} = \operatorname{Sp}\left(B(\lambda)\frac{dA(\lambda)}{d\lambda}\right),$$

$$B(\lambda) = \overline{\Delta}(\lambda) \left\| \begin{matrix} \alpha(\lambda)P^{-1}(\lambda) + M(\lambda) & -P^{-1}(\lambda)u(\lambda) \\ -v(\lambda)P^{-1}(\lambda) & 1 \end{matrix} \right\|,$$

$$M(\lambda) = P^{-1}(\lambda)u(\lambda)v(\lambda)P^{-1}(\lambda),$$

$$\alpha(\lambda) = a_{nn}(\lambda) - v(\lambda)P^{-1}(\lambda)u(\lambda), \qquad (12)$$

$$\frac{dP^{-1}(\lambda)}{d\lambda} = -P^{-1}(\lambda)\frac{dP(\lambda)}{d\lambda}P^{-1}(\lambda),$$

$$\frac{d\overline{\Delta}(\lambda)}{d\lambda} = \overline{\Delta}(\lambda)\operatorname{Sp}\left(P^{-1}(\lambda)\frac{dP(\lambda)}{d\lambda}\right),$$

where $B(\lambda)$ is the adjugate matrix for the matrix $A(\lambda)$, obtained from (10) with the aid of relation (11), with initial conditions

$$\lambda = \lambda_i', \qquad \Delta(\lambda) = \Delta', \qquad P^{-1}(\lambda) = P_0^{-1}, \qquad \overline{\Delta}(\lambda) = \overline{\Delta}'.$$

Here, as the initial value Δ' for the determinant $\Delta(\lambda)$ we take its last value $\Delta(\lambda_i')$, obtained in integrating system (9). The initial matrix P_0^{-1} and the value $\overline{\Delta}'$ of the determinant $\overline{\Delta}(\lambda)$ are found from the formulas

$$P_0^{-1} = Q(\lambda_i') - \frac{r(\lambda_i')q(\lambda_i')}{a_{nn}(\lambda_i')}, \qquad \overline{\Delta}' = \alpha_{nn}(\lambda_i')\Delta(\lambda_i').$$

in which the elements of the inverse matrix are used

$$A^{-1}(\lambda_i') = \begin{Vmatrix} Q(\lambda_i') & r(\lambda_i') \\ q(\lambda_i') & \alpha_{nn}(\lambda_i') \end{Vmatrix},$$

obtained by integrating system (9).

Here $Q(\lambda_i')$ is a matrix, $q(\lambda_i')$ a row, $r(\lambda_i')$ a column, and $\alpha_{nn}(\lambda_i')$ a number.

After system (12) has been integrated on the interval $\lambda_i' \leq \lambda \leq \lambda_i''$, we again return to system (9) and continue to integrate it numerically on the interval $\lambda_i'' \leq \lambda \leq \lambda^*$, taking as the initial value for $\Delta(\lambda)$ its last value $\Delta(\lambda_i'')$, obtained when integrating system (12), and determining the initial matrix for $A^{-1}(\lambda)$ by formula (10) for $\lambda = \lambda_i''$.

3°. The proposed method may also be applied to the computation of determinants of constant matrices. In this case the constant n-dimensional matrix C is represented in the form of the sum of two matrices C_0 and C_1 so that the matrix C_0^{-1} and the value of the determinant $|C_0|$ can be easily found. Then the matrix

$$C_\lambda = C_0 + \lambda C_1$$

for $\lambda = 1$ coincides with the original matrix C, while for $\lambda = 0$ it has the inverse C_0^{-1} and the determinant value $|C_\lambda|$ equal to the determinant value $|C_0|$.

Proceeding with the matrix C_λ analogously to what was set out in 1°, we obtain for the values of the determinant $|C_\lambda|$ the following system of equations, analogous to system (9):

$$\frac{d|C_\lambda|}{d\lambda} = |C_\lambda|\,\mathrm{Sp}(C_\lambda^{-1}C_1),$$

$$\frac{dC_\lambda^{-1}}{d\lambda} = -C_\lambda^{-1} C_1 C_\lambda^{-1}.$$

We integrate this system numerically on the interval $0 \leq \lambda \leq 1$. As the initial conditions we take the values of $|C_\lambda|$ and C_λ^{-1} corresponding to $\lambda = 0$. The value of the determinant $|C_\lambda|$ for $\lambda = 1$ will be the desired value of the determinant of the matrix C.

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Note: Figure translations are in progress. See original paper for figures.