# MATHEMATICS

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## Abstract

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# AN A PRIORI ESTIMATE OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR AN ELLIPTIC OPERATOR WITH DISCONTINUOUS COEFFICIENTS

*(Presented by Academician S. L. Sobolev on 19 XI 1959)*

Let there be given a certain open $N$-dimensional domain $g$ with boundary manifold $\Gamma_2$, and inside it an $(N-1)$-dimensional geometrically closed surface $\Gamma_1$, dividing the domain $g$ into subdomains $g_1$ and $g_2$.

Consider in the closed domain $(g+\Gamma_2)$ the following Dirichlet problem:

$$L_1 u = 0 \quad \text{in the domain } g_1;$$

$$L_2 u = 0 \quad \text{in the domain } g_2;$$

$$[u]\big|_{\Gamma_1} = 0, \qquad \left[\frac{\partial u}{\partial \nu}\right]\bigg|_{\Gamma_1} = k, \qquad u\big|_{\Gamma_2} = 0. \tag{1}$$

Here

$$L_{lu} = \sum_{i,j=1}^{N} a_{ij}^{(l)}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^{(l)}(x)\frac{\partial u}{\partial x_i} + c^{(l)}(x)u \quad (l=1,2) \tag{2}$$

is a linear differential operator of elliptic type defined in the domain $g_l$, i.e. such that for $x=(x_1,\dots,x_N)\in g_l$ the conditions

$$a_{ij}^{(l)} = a_{ji}^{(l)}, \qquad \sum_{i,j=1}^{N} a_{ij}^{(l)}\xi_i\xi_j \geq \alpha^{(l)}\sum_{i=1}^{N}\xi_i^2 \quad (\alpha^{(l)} = \text{const} > 0) \tag{3}$$

are satisfied for any real $\xi_1,\xi_2,\dots,\xi_N$; it is also assumed that

$$c^{(l)} \leq 0 \quad \text{in } g_l \quad (l = 1, 2); \qquad [u]\big|_{\Gamma_1} = u\big|_{x\to\Gamma_1 - 0} - u\big|_{x\to\Gamma_1+0};$$

$$\left[\frac{\partial u}{\partial \nu}\right]\bigg|_{\Gamma_1} \equiv \frac{\partial u}{\partial \nu_1}\bigg|_{x\to\Gamma_1-0} + \frac{\partial u}{\partial \nu_2}\bigg|_{x\to\Gamma_1+0},$$

where $\dfrac{\partial}{\partial \nu_l}$ denotes differentiation in the direction of the conormal, equal to

$$\sum_{i,j=1}^{N} a_{ij}^{(l)} \cos(n^{(l)}, x_j) \frac{\partial}{\partial x_i},$$

and the symbols $\Gamma_1 - 0$ and $\Gamma_1 + 0$ mean that limiting values are taken, respectively, from the inner and from the outer side of the surface $\Gamma_1$ with respect to the domain $g_1$; $k$ is a given function defined on $\Gamma_1$.

We assume that $L_1 \not\equiv L_2$, i.e. that the limiting values of the coefficients of the operators on the boundary $\Gamma_1$ do not coincide identically; thus problem (1) is the Dirichlet problem for an elliptic operator with discontinuous coefficients.

Everywhere in what follows we assume that the coefficients of the operators $L_l$, $a_{ij}^{(l)}(x)$, $b_i^{(l)}(x)$, and $c^{(l)}(x)$ $(l = 1, 2)$, belong to the class* $C^{(0,\mu)}$ $(\mu > 0)$, and that the surfaces $\Gamma_1$ and $\Gamma_2$ belong to the Lyapunov class.

We prove that for any solution of problem (1) belonging to the class $C^{(0)}$ in the closed domain $(g + \Gamma_2)$, to the class $C^{(1)}$ in the closed domains $(g_1 + \Gamma_1)$ and $(g_2 + \Gamma_1 + \Gamma_2)$, and to the class $C^{(2)}$ in the open domains $g_1$ and $g_2$, the following a priori estimate is valid:

$$\max_{x\in(g+\Gamma_2)} |u(x)| \leq C \max_{x\in\Gamma_1} |k(x)|. \tag{4}$$

Here the constant $C$ depends only on the coefficients of the operators $L_l$ and on the form of the domains $g_1$ and $g_2$; moreover, for all elliptic operators with uniformly bounded Hölder coefficients $a_{ij}^{(l)}$, $b_i^{(l)}$, and $c^{(l)}$, and with uniformly bounded values of

$$\frac{1}{a^{(l)}} \qquad (l = 1, 2),$$

the constant $C$ may be taken to be one and the same.

1°. **Lemma 1 (Giraud).** Let $g$ be a bounded $N$-dimensional domain with boundary $\Gamma$ belonging to the Lyapunov class; let $u(x)$ be a solution of the equation $Lu = f$ ($L$ is any of the operators $L_l$, see (2)), continuous and not identically constant in $(g + \Gamma)$, regular in $g$.

If $c \leq 0$, $f \leq 0$ $(f \geq 0)$, $\min_{x\in\Gamma} u \leq 0$ $(\max_{x\in\Gamma} u \geq 0)$, then for each point $y$ on $\Gamma$ at which the function $u(x)$ attains its minimal (maximal) value, and for each ray $l$ issuing from the point $y$ such that $\cos(l, n) < 0$ ($n$ is the inner normal to $\Gamma$), there exists a positive constant $\gamma$ such that, for $x \in l$ and sufficiently small $r_{xy}$ ($r_{xy}$ is the distance between the points $x$ and $y$),

$$u(x) - u(y) > \gamma r_{xy} \quad (< -\gamma r_{xy}).$$

For the proof of Lemma 1 see ([1]), p. 14, or ([2]); see also ([3]).

2°. First of all we establish estimate (4) for the case when $c^{(l)} \equiv 0$ ($l = 1, 2$).

**Lemma 2.** Let the boundary $\Gamma_1$ of the domain $g_1$ belong to the Lyapunov class, and let the coefficients of the operator $L_1$ belong in the domain $(g_1 + \Gamma_1)$ to the class $C^{(0,\mu)}$.

Then there exists a function $\xi(x)$ satisfying the following conditions:

1) $\xi(x) \in C^{(2)}$ in the open domain $g_1$ and satisfies in $g_1$ the equation

$$L_1 \xi(x) = 0.$$

2) $\xi(x) \in C^{(1)}$ in the closed domain $(g_1 + \Gamma_1)$ and on the boundary $\Gamma_1$ is equal to $C = \text{const} > 0$.

3) On the boundary $\Gamma_1$,

$$\partial \xi(x)/\partial \nu < 0, \qquad |\partial \xi(x)/\partial \nu| \geq 1.$$

**Proof.** Consider in the domain $(g_1 + \Gamma_1)$ the following Dirichlet problem:

$$L_1 \bar{\xi}(x) = 0 \quad \text{in the domain } g_1;$$

$$\bar{\xi}(x)\big|_{\Gamma_1} = \bar{C} > 0 \quad (\bar{C} \text{ an arbitrary constant}).$$

Since the boundary surface $\Gamma_1$ belongs to the Lyapunov class, by a result of Giraud ([4]) this problem has a unique solution with first derivatives continuous in the closed domain $(g_1 + \Gamma_1)$

* A function $f(x)$, defined in a bounded closed $N$-dimensional domain $T$, belongs in this domain to the class $C^{(k,\mu)}$ ($C^{(k)}$), if its derivatives of order $k$ satisfy a Hölder condition with exponent $\mu$ in $T$ (are continuous). A function $f(x)$, defined in an open domain $C$, belongs in this domain to the class $C^{(k,\mu)}$ ($C^{(k)}$), if it belongs to this class in every bounded closed domain contained in $C$.

negative. Hence, for the solution on $\Gamma_1$ the quantity $\partial\bar{\xi}/\partial\nu$ is defined and continuous. Since $c^{(1)}(x) \neq 0$ in the domain $g_1$, $\bar{\xi}(x)$ is not constant in $(g_1 + \Gamma_1)$. By Hopf' s theorem, $|\bar{\xi}(x)| < \bar{C}$ for $x \in g_1$.

Applying now Lemma 1, one may assert that $\partial\bar{\xi}/\partial\nu < 0$ everywhere on $\Gamma_1$, and then, by virtue of the continuity of $\partial\bar{\xi}(x)/\partial\nu$ and the closedness of the set of boundary points (the boundary $\Gamma_1$), there is a number $\alpha > 0$ such that $|\partial\bar{\xi}/\partial\nu| > \alpha$ on $\Gamma_1$.

Setting $\zeta(x) = \dfrac{1}{\alpha}\bar{\xi}(x)$, we obtain the required function.

**Theorem.** Let $K_0$ denote $\max\limits_{x\in\Gamma_2}|k(x)|$; let $U_0$ denote $\max\limits_{x\in(g+\Gamma_2)}|u(x)|$. Let the boundary surfaces $\Gamma_1$ and $\Gamma_2$ belong to Lyapunov' s class, and let the coefficients of the operator $L_1$ (respectively $L_2$) belong in the domain $(g_1+\Gamma_1)$ (respectively $(g_2+\Gamma_1+\Gamma_2)$) to the class $C^{(0,\mu)}$.

Then for every solution of problem (1) belonging to the class $C^{(0)}$ in the closed domain $(g+\Gamma_2)$, to the class $C^{(1)}$ in the closed domains $(g_1+\Gamma_1)$ and $(g_2+\Gamma_1+\Gamma_2)$, and to the class $C^{(2)}$ in the open domains $g_1$ and $g_2$, the estimate

$$U_0 \leq CK_0. \tag{5}$$

is valid.

**Proof.** We note that it is sufficient to establish the estimate

$$\max_{x\in(g_1+\Gamma_1)}|u(x)| \leq CK_0, \tag{6}$$

since, by the maximum principle, formula (5) will follow from this. Denote by $\sigma_1$ and $\sigma_2$ two functions defined in the domain $(g_1+\Gamma_1)$:

$$\sigma_1 = K_0\zeta - u, \qquad \sigma_2 = K_0\zeta + u,$$

where $\zeta$ is the function from Lemma 2; $u(x)$ is the solution of problem (1).

We shall show that $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$ in the domain $(g_1+\Gamma_1)$. We carry out the argument, for example, for the function $\sigma_1$. Suppose the contrary, i.e. let $\sigma_1 < 0$ at some point of the domain $(g_1+\Gamma_1)$. Since $L_1\sigma_1 = K_0L_1\zeta - L_1u = 0$ in the domain $g_1$, the negative minimum value of the function $\sigma_1$ is attained on the boundary $\Gamma_1$ (by the maximum principle). Since the function $\zeta(x)$ is, by construction, equal to a constant on $\Gamma_1$, at the point where $\sigma_1$ attains its absolute negative minimum the function $u(x)$ has an absolute positive maximum. Taking into account the conditions $[u]_{\Gamma_1} = 0$, $u|_{\Gamma_2} = 0$ and applying Lemma 1 to the function $u(x)$ in the domains $g_1$ and $g_2$, we conclude that at that point of the boundary $\Gamma_1$, where $u(x)$ attains its maximum positive value,

$$\partial u/\partial\nu_1 < 0$$

and

$$\partial u/\partial \nu_2 < 0.$$

Since $\partial u/\partial \nu_1 + \partial u/\partial \nu_2 = k$, at this point

$$|\partial u/\partial \nu_l| < |k| \quad (l = 1, 2).$$

By Lemma 1, at the point of the absolute negative minimum of the function $\sigma_1(x)$ the relation

$$\frac{\partial \sigma_1}{\partial \nu_1} = K_0 \frac{\partial \zeta}{\partial \nu_1} - \frac{\partial u}{\partial \nu_1} > 0$$

must hold, but this is impossible, since, by Lemma 2, $\partial \zeta/\partial \nu_1 < 0$, $|\partial \zeta/\partial \nu_1| \geq 1$, and, as has just been proved, $|\partial u/\partial \nu_1| < |k| \leq K_0$.

The contradiction obtained proves that $\sigma_1 \geq 0$ everywhere in the domain $(g_1 + \Gamma_1)$. Similarly one establishes that $\sigma_2 \geq 0$. Therefore $|u| \leq |\zeta| K_0$ in the domain $(g_1 + \Gamma_1)$. By virtue of the maximum principle applied to the function $\zeta(x)$, we finally obtain

$$\max_{x \in (g_1 + \Gamma_1)} |u(x)| \leq C K_0,$$

as was required to prove.

3°. We now consider the case when $C^{(l)} = 0$ in the domain $g_l$ $(l = 1, 2)$. In this case we define the function $\zeta(x)$ in the domain $g_2$ by means of the conditions

$$L_2 \zeta(x) = 0 \quad \text{in } g_2, \qquad \zeta(x)|_{x \in \Gamma_1} = C, \qquad \zeta(x)|_{x \in \Gamma_2} = 0.$$

For the function $\zeta(x)$ thus defined in the domain $(g_2 + \Gamma_1 + \Gamma_2)$, all the conditions of Lemma 2 are fulfilled. Carrying out

the proof of the theorem for the domain $(g_2 + \Gamma_1 + \Gamma_2)$, we obtain the required estimate (5).

**Remark 1.** If the domain $g$ is divided by means of $n - 1$ boundary surfaces of Lyapunov class into $n$ nonintersecting domains, then for the solutions of the corresponding Dirichlet problem estimate (5) is valid, in which $K_0$ is to be understood as

$$\max_i \{K_0(g_1), \dots, K_0(g_i), \dots, K_0(g_n)\}.$$

For the proof it suffices to carry out all the arguments for domains on whose boundary $u(x)$ attains an absolute positive maximum and an absolute negative minimum.

**Remark 2.** Estimate (5) is uniform with respect to elliptic operators with uniformly bounded Hölder coefficients $a_{ij}^{(l)}, b_i^{(l)}, c^{(l)}$ and uniformly bounded values of

$$\frac{1}{\alpha^{(l)}} \quad (l = 1, 2).$$

Indeed, the constant $C$ entering estimate (5) depends only on the properties of the function $\zeta(x)$, and for $\zeta(x)$ the assertion formulated in the remark is obvious.

**Remark 3.** It follows from estimate (5) that problem (1) is stable with respect to the function $k$.

**Remark 4.** We have restricted ourselves to considering problem (1), since the general Dirichlet problem for an elliptic operator with discontinuous coefficients:

$$\begin{aligned} L_1 u &= -f_1 \quad \text{in the domain } g_1; \\ L_2 u &= -f_2 \quad \text{in the domain } g_2; \end{aligned} \tag{7}$$

$$[u]\big|_{\Gamma_1} = \varphi, \qquad \left[\frac{du}{d\nu}\right]\bigg|_{\Gamma_1} = \psi, \qquad u\big|_{\Gamma_2} = \chi,$$

where $\varphi, \psi, \chi, f_1$ and $f_2$ are some prescribed functions, can be reduced to problem (1).

Indeed, taking an arbitrary sufficiently smooth function $p$, consider the problem

$$\begin{aligned} L_1 \tilde{u} &= -f_1 && \text{in the domain } g_1; \\ L_2 \tilde{u} &= -f_2 && \text{in the domain } g_2; \end{aligned}$$
$$\tilde{u}\big|_{\Gamma_1,\, x\to\Gamma_1 - 0} = p, \qquad \tilde{u}\big|_{\Gamma_1,\, x\to\Gamma_1 + 0} = p + \varphi, \qquad \tilde{u}\big|_{\Gamma_2} = \chi.$$

The solution $\tilde{u}$ of this problem certainly exists. Representing the solution of the general problem (7) in the form $u = \tilde{u} + \tilde{\tilde{u}}$, we obtain for $\tilde{\tilde{u}}$ problem (1).

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