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Soviet-era science, translated into English

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1960

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**Abstract**

**Full Text**

**A. A. SOKOLOV**

**THE CLOCK PARADOX IN THE MOTION OF CHARGED PARTICLES IN A MAGNETIC FIELD**

*(Presented by Academician N. N. Bogolyubov, 20 XI 1959)*

As is known, in order to analyze the clock paradox, it is necessary to require that the particle, either along the whole path or at least on its separate segments, move with acceleration. This can be done, for example, by considering the motion according to the general theory of relativity <sup>(1,2)</sup>. On the other hand, remaining within the framework of the special theory of relativity, we can describe the accelerated motion of a negatively charged particle, for example an electron in an electromagnetic field, by the relativistically invariant equation (see, for example, <sup>(3)</sup>):

$$m_0 \ddot{x}_\mu = -\frac{e}{c} \dot{x}_\nu H_{\mu\nu}, \tag{1}$$

where the four-dimensional coordinate is  $x_\mu = \mathbf{r}, ict$ , and the magnetic field  $\mathbf{H}$  and electric field  $\mathbf{E}$  are related to the tensor  $H_{\mu\nu}$  by the relations  $H_z = H_{12}$ ,  $iE_x = H_{41}$ , etc. In equation (1), dots denote derivatives with respect to the proper time  $s$ . It should also be noted that in this equation we have neglected the radiation reaction.

We shall use equation (1) to study the clock paradox in the relativistic motion of a charged particle in a betatron. For this purpose, in the plane  $z = 0$  we choose the magnetic field in the form (see, for example, <sup>(4)</sup>)

$$H_x = H_y = 0, \tag{2}$$

$$H_z(r, t) = F(t) \left[ r^{-q} + \pi \frac{1-q}{2-q} R^{2-q} \delta(x) \delta(y) \right]. \tag{3}$$

In order for the motion to be stable, it is necessary that the exponent of the magnetic-field decrease lie within the limits  $0 < q < 1$ . The second term in the brackets on the right-hand side of equality (2) has been added so that, on the equilibrium orbit ( $z = 0, r = R$ ), the Widerøe condition ( $H = \frac{1}{2} \overline{H}$ ) be satisfied. Then, if at the initial instant of time a resting electron is on the equilibrium orbit, then, for an arbitrary increase with time of the magnetic field, i.e. of the function  $F(t)$ , the electron will move along the equilibrium orbit\*.

As is seen from (3), the value of the magnetic field on the equilibrium orbit is equal to

$$H_z(R, t) = F(t)R^{-q}. \quad (4)$$

Since  $H_z$  depends on time, then, along with the magnetic field, there must also appear an electric field, which will accelerate the rotation of the electron.

\* If the initial velocity is different from zero, then we can always choose such an initial magnetic field that will keep the particle on the equilibrium orbit. It is important in this case that the velocity be directed tangentially and that the initial  $r$  be equal to  $R$ .

With the aid of Maxwell's equations it is easy to show that the values of the components of the electric field on the equilibrium orbit will be determined by the relations

$$E_z = 0, \quad E_x = \frac{yR^{-q}}{c}F'(t), \quad E_y = -\frac{xR^{-q}}{c}F'(t), \quad (5)$$

where primes denote derivatives with respect to time  $t$ .

Substituting (4) and (5) into (1) and making the substitution  $x + iy = re^{i\varphi}$ , where  $\varphi$  is the polar angle, we find, for the initial conditions  $z = 0$ ,  $\dot{\varphi} = 0$ ,  $r = R$ , the following equations describing the motion of the electron:

$$\dot{\varphi} = \frac{eH}{m_0c}; \quad (6)$$

$$r = R = \text{const}, \quad z = 0; \quad (7)$$

$$\dot{t} = \frac{1}{\sqrt{1 - \beta^2}}, \quad (8)$$

where  $\beta$  is the velocity of the electron. Since equation (8) is a consequence of all four equations (1), the case  $\mu = 4$  is taken into account in it automatically.

From equation (8) we find

$$s - s_0 = \int_{t_0}^t \sqrt{1 - \beta^2} dt. \quad (9)$$

Next, substituting here the value of the quantity  $\beta$ , which may be found from the equality

$$\dot{\varphi} = \frac{c}{R} \frac{\beta}{\sqrt{1 - \beta^2}} \quad (10)$$

and relation (6), we obtain:

$$s - s_0 = \int_{t_0}^t \frac{dt}{\sqrt{(eRH/m_0c^2)^2 + 1}}. \quad (11)$$

It is evident from this that if the magnetic field is constant ( $F = \text{const}$ ), then the proper time  $s$  and the observer's time  $t$  will be related by the relation, known from the theory of relativity (for motion with constant velocity),

$$s = \sqrt{1 - \beta^2} t. \quad (12)$$

If, however, the magnetic field increases according to the linear law  $F = Bt$ , then

$$s = \frac{1}{\alpha} \ln(\alpha t + \sqrt{1 + \alpha^2 t^2}), \quad (13)$$

where

$$\alpha = \frac{eR^{1-q}}{m_0c^2} B. \quad (14)$$

In deriving formulas (12) and (13) we shall put, at the beginning of the motion,  $s_0 = t_0 = 0$ .

Solving equation (13) with respect to the quantity  $t$ , we obtain

$$t = \frac{1}{\alpha} \text{sh } \alpha s. \quad (15)$$

Taking into account that the quantity  $\alpha t$  is connected with the instantaneous velocity by the relation

$$\alpha t = \frac{\beta}{\sqrt{1 - \beta^2}}, \quad (16)$$

we can represent expression (13) in the form:

$$s = \frac{t}{\beta} \sqrt{1 - \beta^2} \ln \frac{1 + \beta}{\sqrt{1 - \beta^2}} > t \sqrt{1 - \beta^2}. \quad (17)$$

The latter expression is a generalization of formula (12) to the case of motion of a particle with variable velocity, corresponding to an increase of the magnetic field proportional to time.

Let us choose the magnetic field in such a way that it accelerates an electron initially at rest ( $\beta_0 = 0$ ) to some maximum velocity  $\beta_1$ :

$$F(t) = Bt, \quad 0 \leq t \leq t_1, \quad (18a)$$

then, during the time  $t_2$ , maintains this uniform rotation:

$$F(t) = Bt_1 = \text{const}, \quad t_1 \leq t \leq t_1 + t_2, \quad (18b)$$

and, finally, during the time  $t_3 = t_1$ , brings this particle to rest again:

$$F(t) = B(2t_1 + t_2 - t), \quad t_1 + t_2 \leq t \leq t_1 + t_2 + t_3. \quad (18c)$$

Substituting these values of  $F(t)$  into formula (11) and taking into account the solutions (12) and (13), we find the following relations between the corresponding values of the proper time and the observation time:

$$S = s_1 + s_2 + s_3 = \frac{2t_1\sqrt{1-\beta_1^2}}{\beta_1} \ln \frac{1+\beta_1}{\sqrt{1-\beta_1^2}} + t_2\sqrt{1-\beta_1^2}. \quad (19)$$

In the weakly relativistic case ( $\beta_1^2 \ll 1$ ), the latter expression can be represented in the form:

$$S = 2t_1(1 - \frac{1}{6}\beta_1^2) + t_2(1 - \frac{1}{2}\beta_1^2). \quad (20)$$

In the other, ultrarelativistic case ( $1 - \beta_1^2 \ll 1$ ), we shall have:

$$S = 2t_1\sqrt{1-\beta_1^2} \ln \frac{2}{\sqrt{1-\beta_1^2}} + t_2\sqrt{1-\beta_1^2}. \quad (21)$$

However, it should be borne in mind here that in the extreme ultrarelativistic case it is also necessary to take the friction force into account, since losses of energy to radiation must then become appreciable <sup>(5)</sup>.

Conversely, we can find the total time of motion of the particle relative to the observer ( $T = t_1 + t_2 + t_3$ ) as a function of the proper time:

$$T = \frac{2}{a} \text{sh } \alpha s_1 + \frac{s_2}{\sqrt{1-\beta_1^2}}. \quad (22)$$

The formulas obtained for the change of time, apart from clarifying certain general questions connected, for example, with the clock paradox (since we can always formulate the problem in such a way that at the initial and final moments of time the particle is at rest, while  $S < T$ ), can be verified directly experimentally in investigations, for example, of the change in the lifetime of spontaneously decaying charged particles (for example,  $\mu$ - or  $\pi$ -mesons).

Of particular interest here should be the study of the decay probability of particles moving with relativistic, but variable, velocity.

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Received  
20 XI 1959

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*Note: Figure translations are in progress. See original paper for figures.*

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