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**Abstract**

**Full Text**

**A. V. Grekov**

**THE DIRICHLET PROBLEM FOR CERTAIN QUASI-LINEAR PARABOLIC EQUATIONS**

*(Presented by Academician V. I. Smirnov on 18 V 1960)*

The subject of the present note is the proof of existence and uniqueness theorems for the solution of the Dirichlet problem for certain equations of parabolic type of the second order containing functions of two independent variables.

In the plane  $(x, t)$  consider a curvilinear trapezoid  $D$ , bounded by the segments of the straight lines  $t = 0, t = T$  ( $T > 0$ ) and by the curves  $x_1 = \varphi_1(t), x_2 = \varphi_2(t)$ , where  $\varphi_1, \varphi_2$  are continuous functions and  $\varphi_1(t) < \varphi_2(t)$  for  $t \in [0, T]$ . The part of the boundary  $D$  consisting of the segment of the straight line  $t = 0$  and the curves  $\varphi_1(t), \varphi_2(t)$ , will be denoted by  $\Gamma$ . Let  $\bar{D}$  be the closure of  $D$ . Suppose, further, that the function  $u(x, t)$  is defined in  $\bar{D}$  and solves the following problem

$$L_{xt}u \equiv u_t - a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u + f(x, t) = 0 \quad \text{for } x, t \in \bar{D} \setminus \Gamma; \quad (1)$$

$$u|_{t=0} = \varphi(x) \quad \text{for } x \in [\varphi_1(0), \varphi_2(0)];$$

$$u|_{x=\varphi_i(t)} = \psi_i(t) \quad \text{for } t \in [0, T]; \quad i = 1, 2. \quad (2)$$

**Lemma.** Suppose there exists in  $\bar{D}$  a continuous solution of problem (1)–(2) with derivatives  $u_t, u_x, u_{xx}$  continuous in  $\bar{D}$ , and that the conditions

$$a(x, t) \geq a_0 = \text{const} > 0 \quad \text{for } x, t \in \bar{D}; \quad (A)$$

$$\{|u(x, t)|, |a(x, t)|, |b(x, t)|, |c(x, t)|, |f(x, t)|\} \leq K_1 \quad \text{for } x, t \in \bar{D}; \quad (B)$$

$$\{|\varphi_i(t)|, |\psi_i(t)|, |\varphi_i'(t)|, |\psi_i'(t)|\} \leq K_2 \quad \text{for } t \in [0, T]; \quad (C)$$

$$|\varphi(x') - \varphi(x'')| \leq K_3|x' - x''|^\alpha \quad \text{for } x', x'' \in [\varphi_1(0), \varphi_2(0)], \quad 0 < \alpha < 1; \quad (D)$$

$$\varphi[\varphi_i(0)] = \psi_i(0) \quad (i = 1, 2), \quad (\text{E})$$

are fulfilled, where  $K_i$  ( $i = 1, 2, 3$ ) are nonnegative constants. Then the inequality

$$|u(x', t') - u(x'', t'')| \leq K(|x' - x''|^\alpha + |t' - t''|^{\alpha/2})$$

$$\text{for } x', t'; x'', t'' \in \bar{D}; \quad (3)$$

holds;  $K$  is a nonnegative constant depending only on  $K_i, \alpha, a_0$ , and the diameter  $D$  of the trapezoid  $D$ .

**Proof.** In the planes  $(x, t), (y, \tau)$  consider, respectively, the sets  $D_1, \bar{D}_1, \Gamma_1$  and  $D_2, \bar{D}_2, \Gamma_2$ , analogous to  $D, \bar{D}, \Gamma$ . In the space  $(x, y, t, \tau)$  construct the direct product  $\bar{D}_3 = \bar{D}_1 \times \bar{D}_2$ .

Let  $\Gamma_3 = (\Gamma_1 \times \bar{D}_2) \cup (\Gamma_2 \times \bar{D}_1)$ . If the point  $(x, y, t, \tau) \in \bar{D}_3 \setminus \Gamma_3$ , then at it we may write

$$L_x u(x, t) = 0, \quad L_y u(y, \tau) = 0. \quad (4)$$

On  $\bar{D}_3$  consider the function

$$v = v(x, y, t, \tau) = |u(x, t) - u(y, \tau)|^{2/\alpha} / [(x - y)^2 + k|t - \tau|]$$

$$= |u_1 - u_2|^{2/\alpha} / \rho \equiv |\Phi|^{2/\alpha} / \rho,$$

where  $k$  is a certain positive constant, whose choice will be determined below. We shall seek  $\max v$  on  $\bar{D}_3$ . Note that when  $x = y, t = \tau > 0, v = 0$ , and when  $x \neq y, t = \tau = 0, v \leq k_3^{2/\alpha}$ . Let  $v$  attain its greatest value at a point  $A(x, y, t, \tau)$ , with  $t \neq \tau$ . Then either  $A \in \bar{D}_3 \setminus \Gamma_3$ , or  $A \in \Gamma_3$ . Suppose, for example, that  $A \in \bar{D}_3 \setminus \Gamma_3$ . Then at it (4) holds, and also

$$v_x = 0, \quad v_y = 0, \quad v_{xx} \leq 0, \quad v_t \geq 0, \quad v_\tau \geq 0, \quad v_{yy} \leq 0, \quad (5)$$

whence it follows that

$$2a(x, t)v_{xx} + a(y, \tau)v_{yy} - (2v_t + v_\tau) \leq 0. \quad (6)$$

In more detail:

$$v_x = 2|\Phi|^{2/\alpha-1}u_{1x}\delta/\alpha\rho - 2(x-y)|\Phi|^{2/\alpha}/\rho^2 = 0, \quad (7)$$

where  $\delta = \operatorname{sgn} \Phi$ ,

$$\begin{aligned} v_{xx} &= 2|\Phi|^{2/\alpha-1}u_{1xx}\delta/\alpha\rho + 2(2-\alpha)|\Phi|^{2/\alpha-2}u_{1x}^2/\alpha^2\rho \\ &- 8(x-y)|\Phi|^{2/\alpha-1}u_{1x}\delta/\alpha\rho^2 + 8(x-y)^2|\Phi|^{2/\alpha}/\rho^3 - 2|\Phi|^{2/\alpha}/\rho^2 \leq 0. \end{aligned}$$

By virtue of (7) and  $(x-y)^2/\rho < 1$ , we may write

$$0 \geq v_{xx} \geq 2|\Phi|^{2/\alpha-1}u_{1xx}\delta/\alpha\rho - \bar{k}|\Phi|^{2/\alpha}/\rho^2, \quad 0 < \bar{k} < 6. \quad (8)$$

Similarly,

$$0 \geq v_{yy} \geq -2|\Phi|^{2/\alpha-1}u_{2yy}\delta/\alpha\rho - \bar{k}|\Phi|^{2/\alpha}/\rho^2. \quad (9)$$

Further:

$$2v_t + v_\tau = 2|\Phi|^{2/\alpha-1}[2u_{1t} - u_{2\tau}]\delta/\alpha\rho - k|\Phi|^{2/\alpha}\rho^2, \quad t > \tau.$$

Combining (6) and applying (4), with the help of (8) and (9) we obtain

$$\begin{aligned} &\{k - \bar{k}[a(x, t) + a(y, \tau)]\}|\Phi|^{2/\alpha}/\rho + \\ &+ \frac{2}{\alpha}|\Phi|^{2/\alpha-1}\delta[b(x, t)u_{1x} + c(x, t)u_1 + f(x, t) - b(y, \tau)u_{2y} - c(y, \tau)u_2 - f(y, \tau)] \leq 0. \end{aligned}$$

Expressing  $u_{1x}$  and  $u_{2y}$  from the equalities  $v_x = v_y = 0$ , we obtain the inequality

$$\{k - \bar{k}[a(x, t) + a(y, \tau)]\}|\Phi|^{2/\alpha}/\rho \leq \bar{M}|\Phi|^{2/\alpha}/\rho + \bar{N},$$

where  $\bar{M}$  and  $\bar{N}$  are easily computable constants depending only on  $K_1, \alpha$ , and the diameter of  $D$ . Choosing now  $k$  sufficiently large, we obtain the desired estimate.

In the remaining possible positions of the point  $A$ , for which  $v$  attains its greatest value in  $\bar{D}_3$ , the function  $v$  is investigated in an analogous way. The lemma is proved.

A function  $F(x, t, u)$ , defined on the set  $G: x, t \in \bar{D}, |u| \leq M$ , will be said to belong to the class  $C_{\gamma, \gamma/2, \lambda}(G)$  ( $0 < \gamma, \lambda < 1$ ), if for any points  $(x', t', u')$ ,  $(x'', t'', u'')$  from  $G$  the inequality

$$|F(x', t', u') - F(x'', t'', u'')| \leq K(|x' - x''|^\gamma + |t' - t''|^{\gamma/2} + |u' - u''|^\lambda),$$

holds, where  $K = \text{const} \geq 0$ .

Let the function  $g(x, t)$  be defined on  $\bar{D}$ . Introduce the following notation <sup>(5)</sup>:

$$H[g] = \sup |g(x', t') - g(x'', t'')| / (|x' - x''|^\gamma + |t' - t''|^{\gamma/2})$$

for  $x', t'; x'', t'' \in \bar{D}$ ;

$$|g|_\gamma^D = \sup_{\bar{D}} |g| + H[g], \quad |g|_{2+\gamma}^D = \sup_{\bar{D}} \left( \sum_{i=0}^2 |D_x^i g| \right) + \sum_{i=0}^2 H[D_x^i g] + |D_t g|_\gamma^D.$$

**Theorem 1.** *Suppose that the following conditions are satisfied:*

1.  $a(x, t, u) \geq 0$ ,  $\partial f / \partial u \geq \beta = \text{const} > 0$  for  $x, t \in \bar{D}$ ,  $|u| < \infty$ .
2.  $a(x, t, u) \geq a_0 = \text{const} > 0$ ,  $\{a, |b|, |f|\} \leq K_1$ ;  $a, b, \partial f / \partial u \in C_{\gamma, \gamma/2, \lambda[G]}$ ,

where  $G : x, t \in \bar{D}$ ,  $|u| \leq M = \max_{\bar{D}} |f(x, t, 0)| / \beta$ ;  $K_1 = \text{const} \geq 0$ .

3. The functions  $\varphi_i(t)$  ( $i = 1, 2$ ), defining the contour, are such that

$$|\varphi_i(t_1) - \varphi_i(t_2)| \leq K_2 |t_1 - t_2|^\delta$$

for  $t_1, t_2 \in [0, T]$ ,  $\delta = \frac{1}{2} \min(\gamma, \alpha\lambda)$ , and  $K_2 = \text{const} \geq 0$ ,  $0 < \alpha < 1$ .

4.  $f[\varphi_1(0), 0, 0] = f[\varphi_2(0), 0, 0] = 0$ .

Then the problem

$$Lu \equiv u_t - a(x, t, u)u_{xx} + b(x, t, u)u_x + f(x, t, u) = 0 \quad \text{for } x, t \in \bar{D} \setminus \Gamma; \quad (1')$$

$$u|_\Gamma = 0 \quad (2')$$

has at least one solution, and moreover

$$|u|_{2+\delta}^D \leq K_3 |f(x, t, 0)|_\gamma^D,$$

where the constant  $K_3$  depends only on  $K, K_1, K_2, M, a_0, \delta$ , and the diameter of  $D$ .

**Proof.** Any possible solution of problem (1')–(2') of the smoothness required in this theorem is such that  $\max_{\bar{D}} |u| \leq M$  (this follows from the maximum

principle), and such that  $|u|_{\alpha}^D \leq K_4$  (this follows from our lemma), where the constant  $K_4$  depends only on  $K_1, K_2, \alpha, a_0, M$ , and the diameter of  $D$ .

On the domain  $\bar{D}$ , consider the Banach space of functions  $v(x, t)$  such that  $|v|_{\alpha}^D < \infty$  and  $v|_{\Gamma} = 0$ . Denote this space by  $A$ . In it take the ball  $|v|_{\alpha}^D \leq M + K_4$ . Define the transformation  $u = u(v)$  as the solution of the problem

$$u_t - a(x, t, v)u_{xx} + b(x, t, v)u_x + u \int_0^v \frac{\partial f(x, t, y)}{\partial y} dy + f(x, t, 0) = 0; \quad (1'')$$

$$u|_{\Gamma} = 0. \quad (2'')$$

Problem (1)–(2) has a solution <sup>(5)</sup>. Concerning the transformation  $u = u(v)$ , one can say: 1)  $|u|_{\alpha}^D \leq M + K_4$ ; 2)  $|u|_{2+\delta}^D \leq K_3 |f(x, t, 0)|_{\gamma}^D$  <sup>(5)</sup>. Hence it is clear that the hypotheses of Schauder's theorem <sup>(6)</sup> on a fixed point of a continuous, completely continuous transformation of the ball in  $A$  into itself are satisfied. The theorem is proved.

**Theorem 2.** *Suppose that the following conditions are satisfied:*

1.  $a(x, t, u) \geq 0$ ;  $h(x, t, u), \partial f / \partial u$  are continuous for  $x, t \in \bar{D}$ ,  $|u| < \infty$ , and  $\partial f / \partial u \geq \beta = \text{const} > 0$  in the indicated domain.
2.  $a(x, t, u) \geq a_0 = \text{const} > 0$  for  $x, t \in \bar{D}$ ,  $|u| \leq M$  (see Theorem 1).
3.  $a, b, f, h, a_u, b_u, f_u, h_u, a_x, h_x, a_t, h_t, a_{xx}, h_{xx} \in C_{\gamma, \gamma/2, \lambda[G]}$ .
4.  $|\varphi'_i(t_1) - \varphi'_i(t_2)| \leq K_1 |t_1 - t_2|^{\gamma/2}$ ,  $(t_1, t_2) \in [0, T]$ ,  $i = 1, 2$ .
5.  $f[\varphi_1(0), 0, 0] = f[\varphi_2(0), 0, 0] = 0$ .

*Then there exists a unique solution of the problem*

$$u_t - a(x, t, u)u_{xx} + b(x, t, u)u_x + h(x, t, u)u_x^2 + f(x, t, u) = 0, \quad x, t \in D \setminus \Gamma;$$

$$u|_{\Gamma} = 0.$$

such that  $|u|_{2+\alpha}^D \leq K |f(x, t, 0)|_{\gamma}^D$ ,  $0 < \alpha < 1$ .

**Proof.** By the substitution  $v = \int_0^u e^{c(p)} dp$ , where  $|u| \leq M$ ,

$$c(p) = \int_0^p \frac{h(x, t, l)}{a(x, t, l)} dl,$$

this problem is essentially reduced to a problem of type (1')–(2').

The following two theorems are proved with the aid of the existence theorems given in <sup>(1,3)</sup> and the estimates given in <sup>(2,4,5)</sup>.

**Theorem 3.** Suppose the following conditions are satisfied:

1.  $a \geq a_0$ ,  $f_u \leq -\beta$  ( $a_0, \beta = \text{const} > 0$ ) for  $x, t \in \bar{D} = [0, 1] \times [0, T]$ ,  $|u|, |u_x| < \infty$ .
2. In the domain  $|u| \leq M = \max_{\bar{D}} |f(x, t, 0, 0)|/\beta$ ,  $|u_x| < \infty$ , there exist continuous  $a_x, f_x, a_{u_x}, f_{u_x}$ , and moreover  $|f|/a \leq K_1 u_x^2 + K_2 u$ ; for sufficiently large  $|u_x|$  the inequality  $|f_x|/|f_u| \leq K_3 |u_x|^{1-\varepsilon} + K_4$  holds, where  $\varepsilon > 0$ ,  $K_i$  ( $i = 1, 2, 3, 4$ ) are nonnegative constants.
3.  $a, f, a_x, f_x, a_t, f_t, a_{u_x}, f_{u_x}, f_u$  satisfy, with respect to the arguments, a Hölder condition with exponent  $0 < \gamma < 1$  and Hölder constants less than  $K_5$  for  $x, t \in \bar{D}$ ,  $|u| \leq M$ ,  $|u_x| \leq M_1$ .
4.  $f(0, 0, 0, 0) = f(1, 0, 0, 0) = 0$ .

Then there exists a unique solution of the problem

$$u_t = a(x, t, u_x)u_{xx} + f(x, t, u, u_x), \quad xt \in \bar{D} \setminus \Gamma;$$

$$u|_{\Gamma} = 0,$$

and, moreover, there is a  $0 < \alpha < 1$  such that  $|u|_{2+\alpha}^D \leq K_6 = \text{const} \geq 0$ .

**Theorem 4.** Suppose the following conditions are satisfied:

1.  $a \geq 0$ ,  $f_u \leq -\beta$  for  $x, t \in \bar{D}$ ,  $|u|, |u_x| < \infty$ ,  $\beta = \text{const} > 0$ .
2.  $a \geq a_0 = \text{const} > 0$ ,  $|f|/a \leq K_1 u_x^2 + K_2 u$ ; for  $|u_x| \gg 0$  the inequality  $|f_x|/|f_u| \leq K_3 |u_x|^{1-\varepsilon} + K_4$  holds for  $x, t \in \bar{D}$ ,  $|u| \leq M$ ,  $|u_x| < \infty$ ,  $\varepsilon > 0$ .
3.  $a_x, f_x, a_u, f_u, a_{u_x}, f_{u_x}$  are continuous with respect to the arguments, for  $x, t \in \bar{D}$ ,  $|u| \leq M$ ,  $|u_x| < \infty$ .
4. All partial derivatives of  $a, f$  with respect to  $x, u, u_x$  up to and including second order satisfy, with respect to the arguments, a Hölder condition with exponent  $0 < \gamma < 1$  for  $x, t \in \bar{D}$ ,  $|u| \leq M$ ,  $|u_x| \leq M_1$ .
5.  $2|a_{u_x}|/a < K_0$ ,  $|a_{u_x u_x}|/a < K_0$ , where  $K_0 = \varepsilon/G(M_1\sqrt{\rho} + 3M_1^2)$  <sup>(2)</sup>, for  $x, t \in \bar{D}$ ,  $|u| \leq M$ ,  $|u_x| \leq M_1$ .
6.  $f(0, 0, 0, 0) = f(1, 0, 0, 0) = 0$ .

Then there exists a unique solution of the problem

$$u_t = a(x, t, u, u_x)u_{xx} + f(x, t, u, u_x), \quad x, t \in \bar{D} \setminus \Gamma.$$

$$u|_{\Gamma} = 0,$$

and, moreover, there is a  $0 < \alpha < 1$  such that  $|u|_{2+\alpha}^D \leq K_5 = \text{const} \geq 0$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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