On Surfaces Carrying \(\infty^2\) Conical Nets

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Abstract

Full Text

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A Peterson surface, i.e. a surface carrying a conical net, admits the parametrization

$$x_i = A_i(u) + B_i(v) \qquad (i = 1, 2, 3, 4). \tag{1}$$

A conical net is generated by the pair of curves

$$x_i = \frac{dA_i}{du}, \qquad x_i = \frac{dB_i}{dv}, \tag{2}$$

formed by the vertices of the cones of the net.

Translation surfaces constitute the special case of Peterson surfaces when the curves generating the net are improper ($A'_4 = B'_4 = 0$). S. Lie ([1]) proved that translation surfaces carrying ∞^1 nets of translation are generated by a pencil of curves of the second order.

The purpose of the present note is to prove the following analogue of Lie' s theorem.

Theorem. *Peterson surfaces carrying ∞^2 conical nets are generated by pairs of curves lying on surfaces of the second order belonging to one pencil; moreover, on each surface of the pencil there are ∞^1 pairs of these curves.*

Proof. In the papers ([2,3]) it was established that the surfaces carrying ∞^2 conical nets are exhausted, up to collineations (real or imaginary), by the following types:

$$1^0.\ zx = ty^n, \qquad 2^0.\ \frac{z}{t} = \operatorname{arc\,tg}\frac{y}{x}. \tag{3}$$

$$3^0.\ x^3 + yt^2 - xzt = 0, \qquad 4^0.\ \frac{z}{t} = \frac{y}{x} + \lg\frac{x}{t},$$

Substituting in the equations of these surfaces, in place of the coordinates, their parametric expressions according to formulas (1), we obtain equations containing the unknown functions $A_i(u), B_i(v)$. The solution of these functional equations may be represented by the following table:

Type	Canonical representations of the Peterson surface
1^0	$\alpha u - \beta v, \quad \frac{\alpha}{v} - \frac{\beta}{u}, \quad \frac{a}{v^n} - \frac{b}{u^n}, \quad au^n - bv^n$
2^0	$a(\sin v - \sin u) - b(\cos v + \cos u), \quad a(\cos u - \cos v) - b(\sin u + \sin v),$ $\alpha(u+v) + \beta(u^2 - v^2), \quad 2\beta(u-v) + 2\alpha$
3^0	$a(u^2 - v^2) + b(u+v), \quad a(u^4 - v^4) + 2b(u^3 + v^3) + \alpha(u^2 - v^2) + \beta(u+v),$ $2a(u^3 - v^3) + 3b(u^2 + v^2) + \alpha(u - v) + \beta, \quad a(u-v) + b$
4^0	$\frac{a}{v} - \frac{b}{u}, \quad \frac{\beta}{u} - \frac{\alpha}{v} + \frac{a}{v}\lg v - \frac{b}{u}\lg u,$ $\beta v - \alpha u + bv\lg v - au\lg u, \quad au - bv$

In case 1°, the curves generating the net:

$$\begin{array}{cccc} \alpha, & \frac{\beta}{u^2}, & \frac{nb}{u^{n+1}}, & nau^{n-1}, \\ \beta, & \frac{\alpha}{v^2}, & \frac{na}{v^{n+1}}, & nbv^{n-1}, \end{array}$$

are situated on the surfaces of the second order of the pencil

$$\alpha\beta zt = abn^2 xy$$

and are determined by the intersection of these surfaces with the surfaces

$$\begin{aligned} a\beta^n zx^n &= b\alpha^n ty^n, \\ b\alpha^n zx^n &= a\beta^n ty^n. \end{aligned}$$

For $ba^n = a\beta^n$, the curves lie on the surface 1° itself and serve as its asymptotic lines (non-rectilinear).

In case 2°, the curves generating the net:

$$\begin{array}{cccc} b\sin u - a\cos u, & -a\sin u - b\cos u, & \alpha + 2\beta u, & 2\beta, \\ a\cos v + b\sin v, & a\sin v - b\cos v, & \alpha - 2\beta v, & -2\beta, \end{array}$$

are situated on the surfaces of the second order of the pencil

$$4\beta^2(x^2+y^2) = (a^2+b^2)t^2$$

and are determined by the intersection of these surfaces with the surfaces

$$\frac{z}{t} = \operatorname{arc\,tg}\frac{y}{x} \pm \left(\frac{\alpha}{2\beta} - \operatorname{arc\,tg}\frac{b}{a}\right).$$

For

$$\frac{\alpha}{2\beta} = \operatorname{arc\,tg}\frac{b}{a},$$

the curves lie on the helicoid 2° itself and serve as its asymptotic lines (non-rectilinear).

The translation nets of the helicoid are obtained for $\beta = 0$.

In case 3° (Cayley surface of the 3rd order), the curves generating the net:

$$\begin{array}{cccc} 2au+b, & 4au^3+6bu^2+2\alpha u+\beta, & 6au^2+6bu+\alpha, & a, \\ 2av-b, & 4av^3-6bv^2+2\alpha v-\beta, & 6av^2-6bv+\alpha, & a, \end{array}$$

are situated on the surfaces of the second order of the pencil

$$a^2(3x^2-2zt) = (3b^2-2a\alpha)t^2$$

and are determined by the intersection of these surfaces with the surfaces

$$a^3(x^3+yt^2-xzt) \pm (ab\alpha - a^2\beta - b^2)t^2 = 0.$$

For

$$ab\alpha - a^2\beta - b^3 = 0,$$

the curves are situated on the surface 3° itself and serve as its asymptotic lines (non-rectilinear).

The translation nets of the Cayley surface are obtained for $a = 0$.

In case 4°, the curves generating the net:

$$\begin{array}{cccc} \dfrac{a}{u^2}, & \dfrac{b\lg u - b - \beta}{u^2}, & -a\lg u - a - \alpha, & a, \\ \dfrac{a}{v^2}, & \dfrac{a\lg v - a - \alpha}{v^2}, & -b\lg v - b - \beta, & b, \end{array}$$

are located on the second-order surfaces of the pencil

$$xz + yt + \left(\frac{\alpha}{a} + \frac{\beta}{b} + 2\right) xt = 0$$

and are determined by the intersection of these surfaces with the surfaces

$$\frac{z}{t} = -\frac{y}{x} + \lg\frac{x}{t} \pm \left(\frac{\alpha}{a} - \frac{\beta}{b} + \lg\frac{b}{a}\right).$$

For

$$\frac{\alpha}{a} - \frac{\beta}{b} + \lg\frac{b}{a} = 0$$

the curves are located on the surface 4° itself and serve as its asymptotic lines (non-rectilinear).

In cases 1° and 4° one of the second-order surfaces of the pencil may be taken as the absolute of Lobachevsky space; consequently, these surfaces realize the hyperbolic analogue of surfaces of translation with ∞^1 nets of translation, in the sense that they carry ∞^1 conjugate nets composed of cylindrical lines (lines of tangency of the surface with a ruled surface formed by Lobachevsky parallels).

In [4] these surfaces were determined by the coefficients of both differential forms. Here we have their equations in finite form.

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Note: Figure translations are in progress. See original paper for figures.