HYDROMECHANICS

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Abstract

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HYDROMECHANICS

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ON THE APPROXIMATION OF THE REYNOLDS EQUATION

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The solution of various hydromechanical problems leads to the equation

$$\frac{\partial}{\partial x}\left(Q^2\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(Q^2\frac{\partial P}{\partial y}\right)=0, \tag{1}$$

where $P = P(x, y)$ is the unknown function, and Q is a given function, often of one variable, $Q = Q(x)$. Among such problems are problems connected with the theory of plane motions of an ideal gas (^{1}), with the theory of filtration (^{2}), and with the hydrodynamic theory of lubrication (^{3}), in which, in particular, equation (1) is known as the Reynolds equation.

One of the methods for solving boundary-value problems for equation (1) consists in approximating the coefficient $Q(x)$ of this equation by such a function that the solutions of equation (1) are expressed in terms of solutions of a simpler equation possessing the property that the boundary-value problem posed for it is solved in an effective (in particular, closed) form (^{1,4,5}). Since the method described below for obtaining the indicated approximating functions is also suitable for equations of hyperbolic type, in what follows we consider the equation (the indices x and y here and below denote differentiation with respect to the corresponding variable)

$$Q\Delta P + 2Q'P_x = 0, \qquad \Delta = P_{xx} \pm P_{yy}, \tag{1'}$$

which, in the elliptic case, is identical with equation (1).

Consider a sequence of equations of type ($1'$)

$$Q_k\Delta P_k + 2Q_k'(P_k)_x = 0, \qquad k = 0, 1, 2, \dots, \tag{1''}$$

differing in the coefficients $Q_k(x)$. We transform equations ($1''$) by the substitution $u_k = Q_{kP}k$ to the form

$$\Delta u_k = f_{ku}k, \qquad f_k = Q''_{kQ}k^{-1}. \tag{2}$$

Suppose that, by means of the transformation already used by Darboux (^{6}),

$$u_k = A_k(x)u_{k-1} + (u_{k-1})_x, \tag{3}$$

we wish to reduce the solution of a subsequent equation (2) to the solution of the preceding one,

$$\Delta u_{k-1} = f_{k-1}u_{k-1}. \tag{2'}$$

Then, substituting (3) into (2) with account of $(2')$, we obtain a linear homogeneous first-order equation for u_{k-1}, which will be satisfied for any u_{k-1} if its coefficients are set equal to zero, i.e., if one sets

$$A_k'' + (f_{k-1} - f_k)A_k + f_{k-1}' = 0, \qquad 2A_k' + f_{k-1} - f_k = 0. \tag{4}$$

Substituting $f_{k-1} - f_k$ from the second relation (4) into the first and integrating once, we arrive at a Riccati equation for A_k, which, if instead of A_k we introduce the function v_k,

$$A_k = -(\ln v_k)', \tag{5}$$

reduces to a linear equation of the second order (c_k is an arbitrary constant)

$$v_k'' = (f_{k-1} + c_k)v_k. \tag{6}$$

At the same time f_k and Q_k, as is not difficult to verify, are related to v_k by the relations

$$f_k = v_k(v_k^{-1})'' - c_k; \tag{7}$$

$$Q_k = [v_k(v_k^{-1})'' - c_k]Q_k, \tag{8}$$

and if f_{k-1}, determined by equality (7), is substituted into equation (6), then it is brought to the form

$$v_k'' = [v_{k-1}(v_{k-1}^{-1})'' + c_k - c_{k-1}]v_k. \tag{6'}$$

Let us now note $(6,7)$ that if two solutions y_1 and y_2 of the equation $y_i'' = [f(x) + a]y_i$, corresponding to two different values a_1 and a_2 of the constant a $(a_1 \neq a_2)$, are found, then a solution of the equation $z'' = [y_1(y_1^{-1})'' + a_2 - a_1]z$

will be $z = y_1(y_2 y_1^{-1})'$. Applying this rule to equation (6′) and to the equation analogous to (6):

$$v''_{k-1} = (f_{k-2} + c_{k-1})v_{k-1}, \tag{6”}$$

we obtain

$$v_k = v^*_{k-1}(v_{k-1}v^{*-1}_{k-1})', \tag{9}$$

where v^*_{k-1} denotes the solution of equation (6″) corresponding to the replacement in it of c_{k-1} by c_k. Applying the same result to equations (6) and (8), we obtain

$$Q_k = v_k(v_{k0}v_k^{-1})',$$

where v_{k0} denotes the solution of equation (6) corresponding to $c_k = 0$. Since comparison of (6′) and (8) shows that $v_{k0} = Q_{k-1}$, it follows that

$$Q_k = v_k(Q_{k-1}v_k^{-1})'. \tag{10}$$

If we now carry out the differentiation indicated in formulas (9)–(10) and use relation (5), then the recurrence formulas for v_k and Q_k are finally rewritten as follows ($k \geq 2$):

$$v_k = v^*_{k-1} + A_{k-1}v^*_{k-1}, \qquad Q_k = Q'_{k-1} + A_{kQ_{k-1}}, \tag{11}$$

where v_1 is found from equation (6)

$$v''_1 - (f_0 + c_1)v_1 = 0, \tag{12}$$

and Q_1 from (11), taking into account that $Q_0 = v_{10}$.

As we see, if for the chosen f_0 (or Q_0) we are able to solve equation (12), then this determines (with the help of only one differentiation operation) the sequence of coefficients $Q_k(x)$ of equations (1″), whose solutions are expressed in terms of solutions of the equation (replacing u_0 by Φ)

$$\Delta\Phi = f_0\Phi. \tag{13}$$

It follows from this that the coefficient Q_0 (or f_0) must be chosen from the condition of solvability both of equation (12) and of the corresponding boundary-value problem for equation (13).

It is necessary to emphasize that the recurrence formulas (11) are valid only when two conditions are satisfied: $c_k \neq c_{k-1}$ and $c_k \neq 0$. If, however, at some step of the calculations one sets $c_k = c_{k-1}$ or $c_k = 0$, then, as follows from (6′) and (8), respectively, we obtain other formulas for v_k and Q_k ($\alpha_k, \beta_k, \gamma_k, \delta_k$ are arbitrary constants):

$$v_k = v_{k-1}^{-1}\left(\alpha_k + \beta_k \int v_{k-1}^2 \, dx\right), \qquad Q_k = v_k^{-1}\left(\gamma_k + \delta_k \int v_k^2 \, dx\right). \tag{11'}$$

In particular, when $c_k = c_{k-1} = 0$, both formulas (11′) are valid, and then $v_k = Q_{k-1}$, i.e.

$$Q_k = Q_{k-1}^{-1}\left(\gamma_k + \delta_k \int Q_{k-1}^2 \, dx\right). \tag{11''}$$

By combining the recurrence formulas (11), (11′), (11″), one can extend the class of coefficients $Q_k(x)$ for which the solutions of equations (1″) or (2) are expressed in terms of solutions of equation (13). It is interesting to note that formula (11″), which corresponds in the method being presented to the particular case $c_k = c_{k-1} = 0$, coincides with the formula obtained by Yu. V. Rudnev[5] by another method.

Although the recurrence formulas (11)–(12) make it possible in principle to determine the sequence of functions $Q_k(x)$ for which the solutions of equations (1″) or (2) are expressed in terms of solutions of equation (13), their practical use already for $k = 2$ is connected with very cumbersome calculations, which hinder their application. Therefore it is of interest to bring the indicated formulas to a form that would not only facilitate the computation of the functions $Q_k(x)$, but would also make it possible, without computing these functions, to judge some of their properties (periodicity, distribution of zeros, singularities, etc.).

Let us denote by w_ν a solution (general or particular) of the equation

$$w'' = (f_0 + c_\nu) w, \tag{14}$$

where c_ν is a constant. Then, successively writing formula (9) for $k = 2, 3, \ldots$ and noting that, by (6) and (14), $v_1 = w_1$, we arrive at the following theorem:

Theorem 1. *Every solution v_k of an equation belonging to the sequence (6′) is expressed in terms of the solutions $w_1, w_2, \ldots, w_k$ of equation (14), corresponding to different values $c_1, c_2, \ldots, c_k$ of the constant c_ν, in the following way:*

$$v_k = D^{-1}_{kD_{k-1}}, \qquad k = 2, 3, \ldots; \tag{15}$$

D_k is the Wronskian determinant composed of the functions $w_1, w_2, \ldots, w_k$.

Theorem 1 is of independent interest, since, generalizing a known result given in Darboux' s monograph[6,7], it makes it possible, bypassing intermediate calculations, to find the solution of a more complicated equation in terms of the solutions of a simpler equation. It can also be used for the approximation of ordinary second-order equations with variable coefficients.

Let us now denote by D_{k0} the Wronskian determinant composed of the functions $w_1, w_2, \dots, w_{k-1}, w_{k0}$, where w_{k0} is a solution of equation (14) corresponding to $c_k = 0$. Then, noting that $Q_{k-1} = v_{k0}$, taking (15) into account and expressing, by means of (3) and the relation $P_k = u_k/Q_k$, the solution P_k of equation $(1'')$ in terms of the solution P_{k-1} of the analogous equation, we arrive at the following main theorem:

Theorem 2. *With the coefficient of equation (1)*

$$Q_{k-1} = D_{k0} D_{k-1}^{-1}, \qquad k = 2, 3, \dots, \tag{16}$$

the general solution of this equation is expressed in terms of the general solution of the equation $\Delta\Phi = f_0\Phi$ *by means of the recurrence formula*

$$P_{k-1} = P_{k-2} + Q_{k-2} Q_{k-1}^{-1} (P_{k-2})_x, \qquad k = 3, 4, \dots, \tag{17}$$

where

$$P_1 = Q_1^{-1}(A_1\Phi + \Phi_x), \qquad A_1 = -(\ln w_1)'. \tag{18}$$

Relying on Theorem 2, it is easy to formulate a theorem analogous in character also for an equation of any other type whose connection with equation (1) is known (for example, for an equation of type (2)).

Let us note that if one starts from Laplace' s equation, which corresponds to $f_0 = 0$, then the solutions w_{k0} of equation (14) will have the form $w_{k0} = a + bx$, where a, b are arbitrary constants, and in this case in the last column of the determinant D_{k0} only the first two elements can be nonzero.

We now turn to the consideration of two examples connected with the hydrodynamic theory of lubrication. We pose the problem of finding such periodic and continuous coefficients $Q(x)$ of equation (1) for which the solution of this equation would be expressible through the solution of the equation $\Delta\Phi = 0$. Putting $f_0 = 0,\ c_1 = -m_1^2,\ w_1 = \cos(m_1 x + n_1),\ w_{20} = a$, by formula (16) we find

$$Q_1 = D_{20} D_1^{-1} = a_1 \operatorname{tg}(m_1 x + n_1).$$

An analogous expression was obtained by another method by G. A. Dombrovsky (^{4}) and used by him for the effective solution of many problems in the theory of plane motions of an ideal gas. Since the coefficient Q_1 has discontinuities, let us compute Q_2. Putting $f_0 = 0,\ c_1 = -m_1^2,\ c_2 = -m_2^2,\ w_1 = \cos m_1 x,\ w_2 = \cos m_2 x,\ w_{30} = a_2$ and again using (16), we obtain

$$Q_2 = D_{30} D_2^{-1} = a_2 (k_2 \operatorname{tg} t - \operatorname{tg} k_2 t)(k_2 \operatorname{tg} k_2 t - \operatorname{tg} t)^{-1},$$

where $m_2 = k_2 m_1,\ m_1 x = t$ has been set. A simple study of the function Q_2 shows that for $k_2 = (N+1)/N$, where N is a natural number, it is continuous and has periodic zeros of second order. In particular, for $N = 1$ $(k_2 = 2)$, $m_1 = 1/2$, we obtain the function

$$Q_2 = a(1 + 3 \operatorname{ctg}^2 \tfrac{1}{2} x)^{-1},$$

which was used in the author' s work (^{8}) for solving the problem of determining the pressure field in a cylindrical bearing with forced lubricant feed.

Finally, if one puts $f_0 = 0,\ c_1 = +m_1^2,\ w_1 = \operatorname{sh}(m_1 x + n_1),\ w_{20} = a$, then

$$Q_1 = D_{20} D_1^{-1} = a_1 \operatorname{cth}(m_1 x + n_1).$$

Passing here to spherical coordinates (θ, φ) by means of the relations $\hat{\theta} = \ln \operatorname{tg} \frac{1}{2}\theta,\ y = \varphi$ (^{9}), we obtain the function

$$h = Q_1^{2/3} = \left[K \left(M + \operatorname{tg}^{2m} \tfrac{1}{2}\theta \right) \left(M - \operatorname{tg}^{2m} \tfrac{1}{2}\theta \right)^{-1} \right]^{2/3},$$

which can be used to solve the problem of determining the pressure field in a spherical bearing (^{10}), since this function, with a proper choice of the constants K, M, m (introduced instead of a_1, m_1, n_1), is close to the clearance h of the mentioned bearing.

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Note: Figure translations are in progress. See original paper for figures.

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