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Abstract

Full Text

MATHEMATICS

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ON SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH EXPLICIT PERIODIC DEPENDENCE ON THE ARGUMENT

(Presented by Academician S. L. Sobolev on 28 XII 1959)

§ 1. The present article is a further development of the work ⁽¹⁾. The problem solved here is the following. Given functions $u(x, y, z)$ and $v(x, y, z)$, periodic in z with period equal to one, piecewise differentiable in z , analytic in x and y , and with

$$D(u, v)/D(x, y) > 0$$

for all z and all x and y in the domain under consideration. It is required to construct functions $u(x, y, z, t)$ and $v(x, y, z, t)$, periodic in z ,

$$u(x, y, z + 1, t) = u(x, y, z, t), \quad v(x, y, z + 1, t) = v(x, y, z, t), \quad (1)$$

differentiable a sufficient number of times in t , analytic in x and y , having

$$D(u(x, y, z, t), v(x, y, z, t))/D(x, y) \neq 0$$

for all real z and t and for the x and y under consideration, and satisfying the conditions:

- 1) $u(x, y, z, 0) = x, \quad v(x, y, z, 0) = y;$
- 2) $u(x, y, z, 1) = u(x, y, z), \quad v(x, y, z, 1) = v(x, y, z);$
- 3) $u[u(x, y, z, t_1), v(x, y, z, t_1), z + t_1, t_2] = u(x, y, z, t_1 + t_2),$
 $v[u(x, y, z, t_1), v(x, y, z, t_1), z + t_1, t_2] = v(x, y, z, t_1 + t_2).$

Denoting $\tau = z + t$, it is not difficult to show ⁽²⁾ that the required functions, if they exist, must satisfy the system of ordinary equations

$$\frac{du}{dt} = \varphi(u, v, \tau), \quad \frac{dv}{dt} = \psi(u, v, \tau), \quad \frac{d\tau}{dt} = 1, \quad (2)$$

where the functions φ and ψ are periodic in τ :

$$\varphi(u, v, \tau + 1) = \varphi(u, v, \tau), \quad \psi(u, v, \tau + 1) = \psi(u, v, \tau). \quad (3)$$

Theorem 1. *Let two pairs of functions $u_1(x, y, z), v_1(x, y, z)$ and $u_2(x, y, z), v_2(x, y, z)$, satisfying the requirements listed above, coincide for $z = n$, where n is any integer, positive or negative, i.e.*

$$u_1(x, y, n) = u_2(x, y, n), \quad v_1(x, y, n) = v_2(x, y, n),$$

but for $z \neq n$ differ from one another arbitrarily. Then the two systems of differential equations (2), with right-hand sides $\varphi_1(u, v, \tau), \psi_1(u, v, \tau)$ and $\varphi_2(u, v, \tau), \psi_2(u, v, \tau)$ corresponding to these functions, possess solutions which, having identical initial conditions, thereafter coincide for values of t equal to an integral number of periods.

The case considered in the article ⁽¹⁾ belongs to that in which the periodicity in z of one pair, for example $u_2(x, y, z)$ and $v_2(x, y, z)$, reduces to the absence of dependence on this argument. In this case $\varphi_2(u, v, \tau)$ and $\psi_2(u, v, \tau)$ do not depend on τ .

§ 2. Let us consider a method for constructing the required functions (1) and (3) in the form of power series, which is a development of the method of A. N. Korin. Suppose that there exists a solution $(\alpha(z), \beta(z))$ of the system of equations

$$u(\alpha(z), \beta(z), z) = \alpha(z), \quad v(\alpha(z), \beta(z), z) = \beta(z)$$

and that in a neighborhood of it we have the series

$$\begin{aligned} u(x, y, z) &= \alpha(z) + A_{10}(z)(x - \alpha(z)) + A_{01}(z)(y - \beta(z)) + \\ &\quad + A_{20}(z)(x - \alpha(z))^2 + \dots, \\ v(x, y, z) &= \beta(z) + B_{10}(z)(x - \alpha(z)) + B_{01}(z)(y - \beta(z)) + \\ &\quad + B_{20}(z)(x - \alpha(z))^2 + \dots. \end{aligned} \tag{4}$$

The problem has a solution under the condition that the roots ρ_1 and ρ_2 of the equation

$$\begin{vmatrix} A_{10}(z) - \rho & A_{01}(z) \\ B_{10}(z) & B_{01}(z) - \rho \end{vmatrix} = 0 \tag{5}$$

do not depend on z —this is necessary. We shall also assume that the coefficients of (5) are real and $\rho_1 \rho_2 > 0$.

Noting that the functions $u = \alpha(\tau), v = \beta(\tau), \tau = z + t$, are a periodic solution of system (2), it is not difficult to understand that $u(x, y, z, t)$ and $v(x, y, z, t)$ must be sought in the form of the series

$$\begin{aligned} u(x, y, z, t) &= \alpha(z + t) + \alpha_{10}(z, t)(x - \alpha(z)) + \alpha_{01}(z, t)(y - \beta(z)) + \\ &\quad + \alpha_{20}(z, t)(x - \alpha(z))^2 + \dots, \\ v(x, y, z, t) &= \beta(z + t) + \beta_{10}(z, t)(x - \alpha(z)) + \beta_{01}(z, t)(y - \beta(z)) + \\ &\quad + \beta_{20}(z, t)(x - \alpha(z))^2 + \dots. \end{aligned} \tag{6}$$

From formulas (4) and (6) it is clear that, without loss of generality, one may put $\alpha(z) = \beta(z) = 0$. This will correspond to passing to the independent variables $x - \alpha(z)$, $y - \beta(z)$ and to the functions $u(x, y, z, t) - \alpha(z + t)$, $u(x, y, z, t) - \beta(z + t)$. Below we shall assume that such a transition has been carried out. Therefore, for the sought functions $\varphi(x, y, z)$ and $\psi(x, y, z)$ one must write the series

$$\begin{aligned}\varphi(x, y, z) &= \varphi_{10}(z)x + \varphi_{01}(z)y + \varphi_{20}(z)x^2 + \varphi_{11}(z)xy + \varphi_{02}(z)y^2 + \dots, \\ \psi(x, y, z) &= \psi_{10}(z)x + \psi_{01}(z)y + \psi_{20}(z)x^2 + \psi_{11}(z)xy + \psi_{02}(z)y^2 + \dots.\end{aligned}\quad (7)$$

To compute the coefficients $\alpha_{mp}(z, t)$, $\beta_{mp}(z, t)$, $\varphi_{mp}(z)$, and $\psi_{mp}(z)$, we shall use the equations

$$\begin{aligned}\frac{\partial u(x, y, z, t)}{\partial t} &= \varphi(x, y, z) \frac{\partial u(x, y, z, t)}{\partial x} + \psi(x, y, z) \frac{\partial u(x, y, z, t)}{\partial y} + \frac{\partial u(x, y, z, t)}{\partial z}, \\ \frac{\partial v(x, y, z, t)}{\partial t} &= \varphi(x, y, z) \frac{\partial v(x, y, z, t)}{\partial x} + \psi(x, y, z) \frac{\partial v(x, y, z, t)}{\partial y} + \frac{\partial v(x, y, z, t)}{\partial z},\end{aligned}\quad (8)$$

which are easily obtained from (2). Substituting into (8) the series (6) and (7) and equating, on the right and on the left of the equality sign, the coefficients of x and y , we find

$$\begin{aligned}\frac{\partial \alpha_{10}}{\partial t} &= \varphi_{10}(z)\alpha_{10} + \psi_{10}(z)\alpha_{01} + \frac{\partial \alpha_{10}}{\partial z}, & \frac{\partial \beta_{10}}{\partial t} &= \varphi_{10}(z)\beta_{10} + \psi_{10}(z)\beta_{01} + \frac{\partial \beta_{10}}{\partial z}, \\ \frac{\partial \alpha_{01}}{\partial t} &= \varphi_{01}(z)\alpha_{10} + \psi_{01}(z)\alpha_{10} + \frac{\partial \alpha_{01}}{\partial z}, & \frac{\partial \beta_{01}}{\partial t} &= \varphi_{01}(z)\beta_{10} + \psi_{01}(z)\beta_{01} + \frac{\partial \beta_{01}}{\partial z}.\end{aligned}\quad (9)$$

If one equates the coefficients of $x^i y^k$ ($i + k \geq 2$), then one obtains a system of $2(i + k + 1)$ differential equations relating the functions $\alpha_{mp}(z, t)$, $\beta_{mp}(z, t)$, $\varphi_{mp}(z)$, and $\psi_{mp}(z)$, for which the sum of the indices $m + p \leq i + k$. From all these systems of equations, assuming, in view of (1) and (3), the periodicity of the sought coefficients in z , it is poss—

we can determine them if we add here the conditions following from 1), 2), and (4):

$$\begin{aligned}\alpha_{10}(z, 0) &= 1, & \alpha_{01}(z, 0) &= 0, & \alpha_{ik}(z, 0) &= 0, \\ \beta_{10}(z, 0) &= 0, & \beta_{01}(z, 0) &= 1, & \beta_{ik}(z, 0) &= 0, & i + k \geq 2;\end{aligned}\quad (10)$$

$$\alpha_{mp}(z, 1) = A_{mp}(z), \quad \beta_{mp}(z, 1) = B_{mp}(z), \quad m + p \geq 1. \quad (11)$$

Taking into account the conditions (10), we obtain for the expressions $\varphi_{mp}(z)$ and $\psi_{mp}(z)$ in terms of $\alpha_{mp}(z, t)$ and $\beta_{mp}(z, t)$, respectively, the following formulas

$$\varphi_{mp}(z) = \left. \frac{\partial \alpha_{mp}}{\partial t} \right|_{t=0} - \left. \frac{\partial \alpha_{mp}}{\partial z} \right|_{t=0}, \quad \psi_{mp}(z) = \left. \frac{\partial \beta_{mp}}{\partial t} \right|_{t=0} - \left. \frac{\partial \beta_{mp}}{\partial z} \right|_{t=0}. \quad (12)$$

The solution of system (9) that is periodic in z and satisfies the conditions (10) and (11) is equal to

$$\begin{aligned} \alpha_{10}(z, t) &= \rho_1^t \frac{\gamma_1(z)}{\gamma_1(z+t)} \frac{t_{11}(z)t_{22}(z+t)}{d(z+t)} - \rho_2^t \frac{\gamma_2(z)}{\gamma_2(z+t)} \frac{t_{21}(z)t_{12}(z+t)}{d(z+t)}, \\ \alpha_{01}(z, t) &= \rho_1^t \frac{\gamma_1(z)}{\gamma_1(z+t)} \frac{t_{12}(z)t_{22}(z+t)}{d(z+t)} - \rho_2^t \frac{\gamma_2(z)}{\gamma_2(z+t)} \frac{t_{22}(z)t_{12}(z+t)}{d(z+t)}, \\ \beta_{10}(z, t) &= -\rho_1^t \frac{\gamma_1(z)}{\gamma_1(z+t)} \frac{t_{11}(z)t_{21}(z+t)}{d(z+t)} + \rho_2^t \frac{\gamma_2(z)}{\gamma_2(z+t)} \frac{t_{21}(z)t_{11}(z+t)}{d(z+t)}, \\ \beta_{01}(z, t) &= -\rho_1^t \frac{\gamma_1(z)}{\gamma_1(z+t)} \frac{t_{12}(z)t_{21}(z+t)}{d(z+t)} + \rho_2^t \frac{\gamma_2(z)}{\gamma_2(z+t)} \frac{t_{22}(z)t_{11}(z+t)}{d(z+t)}, \\ d(z) &= t_{11}(z)t_{22}(z) - t_{12}(z)t_{21}(z), \end{aligned} \quad (13)$$

where $\gamma_1(z)$ and $\gamma_2(z)$ are arbitrary periodic functions with period equal to one, not vanishing for any z , and the functions $t_{ik}(z)$ are the elements of the matrix

$$\begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix} = \begin{pmatrix} -B_{10}(z) & A_{10}(z) - \rho_1 \\ B_{01}(z) - \rho_2 & -A_{01}(z) \end{pmatrix}, \quad (14)$$

which brings the matrix with elements $A_{10}(z)$, $A_{01}(z)$, $B_{10}(z)$, and $B_{01}(z)$ to diagonal form. For the functions $\varphi_{10}(z)$, $\varphi_{01}(z)$, $\psi_{10}(z)$, and $\psi_{01}(z)$, using (12) we find the expressions

$$\begin{aligned}
\varphi_{10}(z) &= \omega_{10}(z) - \frac{t'_{11}(z)t_{22}(z) - t_{12}(z)t'_{21}(z)}{d(z)} \\
&\quad - \left(\frac{t_{11}(z)t_{22}(z)}{d(z)} \frac{\gamma'_1(z)}{\gamma_1(z)} - \frac{t_{12}(z)t_{21}(z)}{d(z)} \frac{\gamma'_2(z)}{\gamma_2(z)} \right), \\
\psi_{10}(z) &= \sigma_{10}(z) - \frac{t_{11}(z)t'_{21}(z) - t'_{11}(z)t_{21}(z)}{d(z)} - \frac{t_{11}(z)t_{21}(z)}{d(z)} \left(\frac{\gamma'_2(z)}{\gamma_2(z)} - \frac{\gamma'_1(z)}{\gamma_1(z)} \right), \\
\varphi_{01}(z) &= \omega_{01}(z) - \frac{t_{22}(z)t'_{12}(z) - t_{12}(z)t'_{22}(z)}{d(z)} - \frac{t_{12}(z)t_{22}(z)}{d(z)} \left(\frac{\gamma'_1(z)}{\gamma_1(z)} - \frac{\gamma'_2(z)}{\gamma_2(z)} \right), \\
\psi_{01}(z) &= \sigma_{01}(z) - \frac{t_{11}(z)t'_{22}(z) - t_{21}(z)t'_{12}(z)}{d(z)} \\
&\quad - \left(\frac{t_{11}(z)t_{22}(z)}{d(z)} \frac{\gamma'_2(z)}{\gamma_2(z)} - \frac{t_{12}(z)t_{21}(z)}{d(z)} \frac{\gamma'_1(z)}{\gamma_1(z)} \right),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\omega_{10}(z) &= \frac{t_{11}(z)t_{22}(z) \ln \rho_1 - t_{12}(z)t_{21}(z) \ln \rho_2}{d(z)}, & \sigma_{10}(z) &= \frac{t_{11}(z)t_{21}(z)}{d(z)} \ln \frac{\rho_2}{\rho_1}, \\
\omega_{01}(z) &= \frac{t_{22}(z)t_{12}(z)}{d(z)} \ln \frac{\rho_1}{\rho_2}, & \sigma_{01}(z) &= \frac{t_{11}(z)t_{22}(z) \ln \rho_2 - t_{12}(z)t_{21}(z) \ln \rho_1}{d(z)}.
\end{aligned} \tag{16}$$

In the case where ρ_1 and ρ_2 satisfy the inequalities

$$\rho_1 \neq \rho_2, \quad \rho_1 \neq \rho_1^{n-q} \rho_2^q, \quad \rho_2 \neq \rho_1^{n-q} \rho_2^q, \tag{17}$$

where $n \geq 2$, $0 \leq q \leq n$; the further computations of $\alpha_{mp}(z, t)$ and $\beta_{mp}(z, t)$ can be simplified by using the functional equations

$$\begin{aligned}
u[u(x, y, z, t), v(x, y, z, t), z + t, 1] &= u[u(x, y, z, 1), v(x, y, z, 1), z, t], \\
v[u(x, y, z, t), v(x, y, z, t), z + t, 1] &= v[u(x, y, z, 1), v(x, y, z, 1), z, t],
\end{aligned} \tag{18}$$

which follow from 3) with (1) taken into account.

Substituting the series (4) and (6) into (18), equating the coefficients of $x^m y^p$ ($m + p = n \geq 2$) on both sides of the equalities, and regarding the quantities $\alpha_{ik}(z, t)$ and $\beta_{ik}(z, t)$ with $i + k \leq n - 1$ as known, we obtain $2(n + 1)$ linear algebraic equations for the same number of unknown quantities $\alpha_{mp}(z, t)$ and $\beta_{mp}(z, t)$ with a nonzero determinant. The functions $\alpha_{10}(z, t)$, $\alpha_{01}(z, t)$, $\beta_{10}(z, t)$, and $\beta_{01}(z, t)$, however, cannot be found uniquely in this way.

When the conditions (17) are satisfied, the series for $u(x, y, z, t)$ and $v(x, y, z, t)$ can be arranged in powers of ρ_1^t and ρ_2^t :

$$u(x, y, z, t) = u_{10}(x, y, z, t)\rho_1^t + u_{01}(x, y, z, t)\rho_2^t + u_{20}(x, y, z, t)\rho_1^{2t} + \dots,$$

$$v(x, y, z, t) = v_{10}(x, y, z, t)\rho_1^t + v_{01}(x, y, z, t)\rho_2^t + v_{20}(x, y, z, t)\rho_1^{2t} + \dots, \quad (19)$$

where $u_{mp}(x, y, z, t)$ and $v_{mp}(x, y, z, t)$ are functions periodic in t . If in formulas (19) one passes to the limiting linear case ($A_{ik}(z) = B_{ik}(z) = 0$, $i + k \geq 2$), then one obtains the usual formulation of Floquet's theorem.

§ 3. In describing the method for constructing the series (6) and (7), we assumed that the roots (5) do not depend on z , that the sum $\rho_1 + \rho_2$ is real, that $\rho_1 \neq \rho_2$, and that $\rho_1\rho_2 > 0$. However, convergence of these series has so far been proved under more stringent conditions. The following theorem holds, entirely analogous to Theorem 1 of paper (1):

Theorem 2. *Let the real functions $u(x, y, z)$ and $v(x, y, z)$, periodic in z , be expandable in a neighborhood of the point $x = 0$, $y = 0$ into the series (4), where $\alpha(z) = \beta(z) = 0$, and suppose that the roots ρ_1 and ρ_2 of equation (5) do not depend on z and are: 1) either complex conjugate and distinct, 2) or real, subject to the conditions $\rho_1\rho_2 > 0$, (17), and one of the pairs of inequalities*

$$|\rho_1| < 1, \quad |\rho_2| < 1 \quad \text{or} \quad |\rho_1| > 1, \quad |\rho_2| > 1.$$

Then, for any finite t , there exists a neighborhood of the point $x = 0$, $y = 0$ in which the functions $u(x, y, z, t)$ and $v(x, y, z, t)$ are expandable into the series (6) with coefficients periodic in z .

From any neighborhood of $x = 0$, $y = 0$, when the conditions of Theorem 2 are satisfied, the functions $u(x, y, z, t)$, $v(x, y, z, t)$ can be analytically continued into a broader domain, which in paper (1) was called iterative. In the present case, for each z there will be its own domain. To obtain the values of the functions at some point of the named domain, one must use equations (18), in which the unit is replaced by the corresponding integer n , positive or negative, while as x and y one takes the coordinates of those points from the neighborhood where the series certainly converge.

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Note: Figure translations are in progress. See original paper for figures.

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