MATHEMATICS

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Abstract

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THE DIRICHLET PRINCIPLE FOR THE BELTRAMI EQUATION IN A HALF-SPACE

(Presented by Academician S. L. Sobolev, 17 V 1960)

1. The Beltrami equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\mu}{x_n}\frac{\partial u}{\partial x_n} \equiv \Delta u + \frac{\mu}{x_n}\frac{\partial u}{\partial x_n} = 0 \tag{1}$$

has been the subject of numerous investigations ([1]). For $-1 < \mu < 1$ we shall consider for this equation the Dirichlet problem in the following formulation (due to L. D. Kudryavtsev):

I. Find a twice continuously differentiable function $u(x_1, \dots, \dots, x_n) \equiv u(x)$, satisfying equation (1) for $x_n > 0$, having as its trace* for $x_n = 0$ the function $\varphi(x_1, \dots, x_{n-1}) \equiv \varphi(x)$, and possessing the finite weighted integral

$$\int_{E_n^+ (x_n > 0)} x_n^{\mu} (\operatorname{grad} u)^2 \, dX. \tag{2}$$

The main result that we prove is as follows:

Theorem 1. *Problem I has a solution, and moreover a unique one, if and only if*

$$\varphi(x) \in W_2^{1-\frac{1+\mu}{2}}(E_{n-1}).$$

This solution can be obtained by minimizing the functional (2) in the class of all functions from $\widehat{W}^1_{2,\mu}(E_n^+)$ *with trace* $\varphi(x)$.

This theorem complements the corresponding result of L. D. Kudryavtsev. Namely, in papers ([2,3]) the formulated variational principle was justified for $0 \leq \mu < 1$ under more stringent requirements on the boundary function (see also the footnote on p. 762). Our result has, in a certain sense, a final character, by virtue of the simultaneous necessity and sufficiency of the conditions imposed on $\varphi(x)$.

2. We shall give a scheme of the proof of Theorem 1, dwelling in more detail on those points which are new in comparison with ([2,3]).

The set of functions $u(X)$, locally summable in the "upper" half-space E_n^+ ($x_n > 0$), possessing in E_n^+ generalized derivatives $\partial u/\partial x_i$ with finite weighted integral

$$D_{2,\mu}(u) = \int_{E_n^+} x_n^\mu \sum \left(\frac{\partial u}{\partial x_i}\right)^2 dX,$$

whose traces $u(X)|_{x_n=0} = \varphi(x)$ on the hyperplane E_{n-1} are summable with

* The function $\varphi(x)$ is called the **trace** of the function $u(X)$ on the hyperplane E_{n-1} ($x_n = 0$), if for almost all $x \in E_{n-1}$ there exists the limit $\lim_{x_n\to 0} u(x_1, \dots, x_{n-1}, x_n)$, equal to $\varphi(x)$.

with a square-integrable square ($\varphi(x) \in L_2(E_{n-1})$), we shall denote by $W^1_{2,\mu}(E_n^+)$. By introducing the norm

$$\|u\|^2_{\widehat{W}^1_{2,\mu}} = \|\varphi\|^2_{L_2(E_{n-1})} + D_{2,\mu}(u)$$

this set becomes a complete normed space.

From the results of our paper ([4]) it follows that, for any function $u(X) \in \widehat{W}^1_{2,\mu}(E_n^+)$, the trace $\varphi(x)$ exists and is a function in $W_2^{1-\frac{1+\mu}{2}}(E_{n-1})$, i.e.:

$$\text{a) } \varphi(x) \in L_2(E_{n-1}); \qquad \text{b) } \int_{E_{n-1}} dx \int \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{n-2+p-\mu}}\, dy < \infty.$$

Conversely, the membership $\varphi(x) \subset \widehat{W}_2^{1-\frac{1+\mu}{2}}(E_{n-1})$ (the presence in $\varphi(x)$ of properties a), b)) makes it possible to construct a function $u(X) \in \widehat{W}^1_{2,\mu}(E_n^+)$ for which $\varphi(x)$ is the trace ([4]). Hence it follows that the class $\widehat{W}^1_{2,\mu}\{\varphi\}$ of admissible functions in the variational problem under consideration is nonempty.

The existence of a unique minimizing element $U(X)$ for the functional (2), and its membership in $\widehat{W}^1_{2,\mu}\{\varphi\}$, are easily derived from the completeness of the space and the properties of the functional (2) ([2,3,5]). The function $U(X)$ is then analytic ([5]) and is the unique solution of the differential problem I*. To prove uniqueness we prove the following lemma.

Lemma. Let a twice continuously differentiable function $U(X)$ belong to $\widehat{W}^1_{2,\mu}$ and satisfy equation (1). Then $D_{2,\mu}(U, \psi) =$

$$= \int_{E_n^+} x_n^\mu (\operatorname{grad} U \ \operatorname{grad} \psi)\, dX$$

vanishes for any function $\psi(X) \in \widehat{W}^1_{2,\mu}$ with zero trace on E_{n-1}.

First, by integration by parts we establish the formula

$$D_{2,\mu}[U,\psi;\Pi] = \int_\Pi x_n^\mu(\operatorname{grad} U \ \operatorname{grad}\psi)\,dX = \int_S x_n^\mu \psi \frac{\partial U}{\partial n}\,dS,$$

where Π is the "rectangular box" $\{-a < x_i < a,\ i = 1,\ldots,n-1,\ 0 < \delta < x_n < M\}$, and S is its boundary. It is obvious that

$$D_{2,\mu}(U,\psi) = \lim_{\Pi\to E_n^+} D_{2,\mu}[U,\psi;\Pi].$$

The lemma will be proved if we establish the existence of a sequence of rectangular boxes $\{\Pi_j\}$ with boundaries S_j, for which we have

$$\int_{S_j} x_n^\mu \left|\psi \frac{\partial U}{\partial n}\right| dS \to 0, \qquad \text{as } \Pi_k \to E_n^+.$$

Fix the "upper" and "lower" bases of the rectangular boxes Π (δ and M) and consider the integral

$$\int_{E_n^{\delta M}} x_n^\mu \left|\psi \frac{\partial U}{\partial x_i}\right| dX,$$

extended over the layer of space $E_n^{\delta M}$ ($\delta < x_n < M$). From Hölder' s inequality there follows the estimate

$$I^{\delta M} \equiv \int_{E_n^{\delta M}} x_n^\mu \left|\psi \frac{\partial U}{\partial x_i}\right| dX \le \frac{M^{2-\mu}}{1-\mu} \max\{M^{\mu/2}, \delta^{\mu/2}\}\, D_{2,\mu}^{1/2}(U) D_{2,\mu}^{1/2}(\psi).$$

* In papers ([2,3]) the indicated uniqueness is proved under additional requirements on the sought function at infinity.

therefore, necessarily, there must be a sequence x_{ik} (i is the number of the axis, $k = 0, 1, \ldots$) for which

$$\lim_{|x_{ik}|\to\infty} S_i^{\delta M}(x_{ik}) \equiv \lim_{|x_{ik}|\leftarrow\infty} \int_\delta^M dx_n \int_{-\infty}^{\infty} \cdots \int x_n^\mu \left|\psi \frac{\partial U}{\partial x_i}\right|_{x_i = x_{ik}} dx_1 \ldots dx_{n-1} = 0;$$

otherwise the integral $I^{\delta M}$ could not exist. Since the integral over the "lateral" surface Π is majorized by the sum of the integrals

$$\sum_{i=1}^{n} S_i^{\delta M},$$

we obtain

$$\left|D_{2,\mu}\left[U,\psi,E^{\delta M}\right]\right| \leq \int_{E_{n-1}} x_n^\mu \left|\psi\frac{\partial U}{\partial x_n}\right|_{x_n=\delta} dx + \int_{E_{n-1}} x_n^\mu \left|\psi\frac{\partial U}{\partial x_n}\right|_{x_n=M} dx, \quad (3)$$

and it remains only to pass once more to the limit in δ and M.

Estimating each term on the right-hand side of (3) separately, we write

$$\int_{E_{n-1}} x_n^\mu \left|\psi\frac{\partial U}{\partial x_n}\right|_{x_n=\delta} dx \leq \left\{\int_{E_{n-1}} x_n^\mu \left(\frac{\partial U}{\partial x_n}\right)^2_{x_n=\delta} dx\right\}^{1/2} \cdot \left\{\int_{E_{n-1}} x_n^\mu (\psi)^2\big|_{x_n=\delta} dx\right\}^{1/2} =$$

$$= F^{1/2}(\delta)\,\delta^{\mu/2}\left[\int_{E_{n-1}} \left|\psi(x_1,\dots,x_{n-1},\delta)-\psi(x_1,\dots,0)\right|^2 dx\right]^{1/2} \leq$$

$$\leq cF^{1/2}\delta^{1/2}\left[\int_{E_{n-1}} dx \int_0^\delta x_n^\mu \left(\frac{\partial\psi}{\partial x_n}\right)^2 dx_n\right]^{1/2} = F^{1/2}(\delta)O(\delta^{1/2}).$$

In view of the finiteness of the integral $D_{2,\mu}(U)$, one may assert the existence of a sequence $\delta_k \to 0$ for which

$$F(\delta_k) \equiv \int_{E_{n-1}} x_n^\mu \left(\frac{\partial U}{\partial x_n}\bigg|_{x_n=\delta_k}\right)^2 dx \leq \frac{c}{\delta_k}.$$

Therefore, in the passage to the limit along the sequence δ_k, the first term in formula (3) vanishes; the second term is estimated analogously, proceeding from the existence of a sequence $M_k \to 0$ for which

$$F(M_k) \leq \frac{c}{M_k \ln M_k}.$$

The lemma is proved.*

Theorem (uniqueness). The function $u(X)$ is the unique solution of problem I in the class $\widehat{W}^1_{2,\mu}\{\varphi\}$.

Let, besides the solution $U(X)$, there exist another solution $V(X) \in \hat{W}^1_{2,\mu}\{\varphi\}$. Then the difference $U - V = \psi(X)$ will satisfy the conditions of the lemma just proved, and we have

$$D_{2,\mu}(U) = D_{2,\mu}[V + U - V] = D_{2,\mu}(V) + 2D_{2,\mu}(V, U - V) + D_{2,\mu}(U - V) =$$

$$= D_{2,\mu}(V) + D_{2,\mu}(U - V) > D_{2,\mu}(V),$$

which contradicts the minimal property of U.

3. Let us emphasize once again the naturalness with which the classes $W_p^{(r)}$ are used in estimates relating to the solution of the Dirichlet problem in the half-space E_n^+ ($x_n > 0$) for equation (1) with $\mu < 1$; in particular, these estimates are also valid for the solution of problem I.

Theorem. Suppose that the function

$$\frac{\varphi(x)}{(1 + |X|^2)^{\frac{n-\mu}{2}}}$$

is summable in E_{n-1}. Then the function**

$$U_\mu(X) = \pi^{\frac{1-\mu}{2}} \frac{\Gamma\left(\frac{n-\mu}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)} x_n^{1-\mu} \int_{E_{n-1}} \frac{\varphi(y)\, dy}{(|x - y|^2 + x_n^2)^{\frac{n-\mu}{2}}} \tag{4}$$

* The device used in its proof belongs to S. M. Nikol' skii ([6]).

** Formula (4) is given in the work ([7]).

is a solution of equation (1) and has trace function $\varphi(x)$. If, moreover, the function $\varphi(x)$ belongs to $W_p^{(r)}(E_{n-1})$ and r is represented in the form

$$r = \bar{r} - \frac{1 + \alpha}{p},$$

where $\bar{r} \geq 1$ is an integer, and $-1 < \alpha < p - 1$, then the inequalities

$$\int_{E_n^+} x_n^{\alpha + pl} \sum_{\alpha_1 + \dots + \alpha_n = \bar{r} + l} \left| \frac{\partial^{\bar{r}+l} u_\mu}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|^p dX \leq$$

$$\leq C_l \sum \int_{E_{n-1}} dx \int \frac{\left|\varphi^{(\bar{r}-1)}(x) - \varphi^{(\bar{r}-1)}(y)\right|^p}{|x - y|^{n-2+p-\alpha}} dy, \qquad l = 0, 1, \dots, \tag{5}$$

where C_l does not depend on φ, and the sum on the right is taken over all derivatives of order $\bar{r}-1$ of the function φ.

The proof of the first part of the theorem is carried out in the same way as for the ordinary Poisson integral, which is obtained in (4) for $\mu = 0$. We outline the proof of the second part.

First let $\bar{r} = 1$. We compute and estimate the derivatives $\partial U_\mu / \partial x_i,\ i \neq n$:

$$\frac{\partial U_\mu}{\partial x_i} = K(n-\mu) x_n^{1-\mu} \int_{E_{n-1}} \frac{[\varphi(x)-\varphi(y)](x_i - y_i)}{(|x-y|^2 + x_n^2)^{\frac{n-\mu}{2}+1}}\, dy.$$

With the help of Hölder' s inequality, for

$$\varepsilon < 1 + \frac{n-1}{p}$$

we obtain

$$\left|\frac{\partial U_\mu}{\partial x_i}\right| \leq \frac{c}{x_n^{\frac{n+p-1}{p}}} \left(\int_{E_{n-1}} \frac{|\varphi(x)-\varphi(y)|^p\, dy}{\left(1 + \frac{|x-y|^2}{x_n^2}\right)^{\varepsilon p/2}} \right)^{1/p}.$$

Now computing the weighted integral of $\partial U_\mu / \partial x_i$ by changing the order of integration under the additional (noncontradictory) requirement

$$\varepsilon > \frac{p+n-\alpha}{p},$$

we easily obtain

$$\int_{E_n^+} x_n^\alpha \left|\frac{\partial U_\mu}{\partial x_i}\right|^p dX \leq c \int_{E_{n-1}} dx \int_{E_{n-1}} \frac{|\varphi(x)-\varphi(y)|^p\, dy}{|x-y|^{\,n-2+p-\alpha}}.$$

The estimates involving $\partial U_\mu / \partial x_n$ proceed analogously, and we obtain (5) for $\bar{r} = 1,\ l = 0$. In estimating higher derivatives ($l \neq 0$), at each subsequent differentiation we shall be forced to increase the exponent of the weight by p, in order to cancel the singularity arising in the differentiation. For $\bar{r} > 1$ the argument proceeds according to the same scheme, if one first transfers part of the differentiations under the integral sign from the kernel to the function φ.

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REFERENCES

1. A. Weinstein, Proc. Intern. Congr. Math., **3**, 264 (1956).
2. L. D. Kudryavtsev, DAN, **107**, No. 4 (1956).
3. L. D. Kudryavtsev, DAN, **108**, No. 1 (1956).
4. P. I. Lizorkin, DAN, **132**, No. 3 (1960).
5. L. D. Kudryavtsev, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **55** (1959).
6. S. M. Nikol' skii, Izv. AN SSSR, ser. matem., **22**, 599 (1958).
7. M. N. Olevskii, DAN, **64**, No. 6 (1949).

Note: Figure translations are in progress. See original paper for figures.