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Abstract

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On One New Class of Parabolic Systems of Equations

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It is known that, in investigating the question of the correct solvability of the Cauchy problem for systems with coefficients continuously depending on the time variable t, of the form

$$\frac{\partial u_j}{\partial t} = \sum_{k=1}^{N} \sum_{\Sigma r_s \le p_k} a_{jk}^{(r_1 \dots r_n)}(t) \frac{\partial^{r_1+\dots+r_n} u_k}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \qquad (j = 1, \dots, N), \tag{1}$$

an essential role is played by the behavior of the normal fundamental matrix $Q(\alpha, t, t^0)$ of solutions of the adjoint system

$$\frac{dv_j}{dt} = \sum_{k=1}^{N} \sum_{\Sigma r_s \le p_k} a_{jk}^{(r_1 \dots r_n)}(t) (i\alpha_1)^{r_1} \cdots (i\alpha_n)^{r_n} v_k \tag{2}$$

for large real values of the parameters α_s, since whether or not the known condition (A) of I. G. Petrovskii is fulfilled depends on the behavior of this matrix as $\alpha_1^2 + \dots + \alpha_n^2 \to \infty$. Among systems satisfying condition (A), I. G. Petrovskii singled out two principal classes of systems, called hyperbolic and parabolic. In the work (^{2}), S. A. Galpern singled out one more class of systems satisfying condition (A). For systems with constant coefficients, G. E. Shilov (^{3}) considered a number of very broad classes of systems for which condition (A) is fulfilled.

In the present note a new class of systems of the form (1) is singled out, which satisfy condition (A) of I. G. Petrovskii (even a stronger one; see (8) below). The new class extends the class of parabolic systems in the sense of I. G. Petrovskii, and at the same time the basic properties inherent in the former class are preserved in it.

In the further discussion it is more convenient to write the system in vector form

$$\frac{dV}{dt} = A(\alpha, t)V, \tag{3}$$

where $V(\alpha, t) \equiv (v_1(\alpha, t), \dots, v_N(\alpha, t))$,

$$A(\alpha, t) \equiv \left(\sum_{\Sigma r_s \le p_k} a_{jk}^{(r_1 \dots r_n)}(t)(i\alpha_1)^{r_1} \cdots (i\alpha_n)^{r_n} \right). \tag{4}$$

Without loss of generality one may assume that in the system (1) the weights p_j (we shall call the number p_j—the highest order of derivatives with respect to x of the function u_j in (1)—the weight of the function u_j) are arranged so that

$$p_1 = p_2 = \dots = p_{N_1} = h_1;$$

$$p_{N_1+1} = p_{N_1+2} = \dots = p_{N_1+N_2} = h_2;$$

$$p_{N_1+\dots+N_{l-1}+1} = p_{N_1+\dots+N_{l-1}+2} = \dots = p_{N_1+\dots+N_l} (\equiv p_N) = h_l,$$

where $h_1 > h_2 > \dots > h_l$. We shall call the terms

$$a_{jk}^{(r_1 \dots r_n)}(t) \frac{\partial^{r_1+\dots+r_n} u_k}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}$$

in system (1) (respectively, the terms $a_{jk}^{(r_1 \dots r_n)}(t)(i\alpha_1)^{r_1} \cdots (i\alpha_n)^{r_n}$ in matrix (4)) **principal** if for them $\sum r_s = p_k$.

We impose on the matrix $A(\alpha, t)$, and thereby on the coefficients of system (1), the following restrictions:

I. The matrix

$$A^0(\alpha, t) \equiv \left(\sum_{\sum r_s = p_k} a_{jk}^{(r_1 \dots r_n)}(t)(i\alpha_1)^{r_1} \cdots (i\alpha_n)^{r_n} \right), \tag{5}$$

formed only from the principal terms of the matrix $A(\alpha, t)$, has the block-diagonal form

$$A^0 = \begin{pmatrix} A_{11}^0 & & & \\ & A_{22}^0 & & \\ & & \ddots & \\ & & & A_{ll}^0 \end{pmatrix}, \tag{6}$$

where A_{mm}^0 $(m = 1, 2, \dots, l)$ is a square matrix of order N_m, such that

$$A^0_{mm} \equiv \left(\sum_{\sum r_s = h_m} a^{(r_1 \dots r_n)}_{jk}(t)(i\alpha_1)^{r_1} \cdots (i\alpha_n)^{r_n} \right)$$

$$(N_1 + \dots + N_{m-1} + 1 \le j, k \le N_1 + \dots + N_m), \tag{7}$$

and all elements not falling into any one of the squares A^0_{mm} $(m = 1, 2, \dots, l)$ are identically equal to zero. In particular, the case is possible when a single block A^0_{11} fills the entire matrix A^0.

II. All matrices A^0_{mm} $(m = 1, 2, \dots, l)$ in (6) satisfy the condition (B) of I. G. Petrovskii, i.e., for all real $\alpha_1, \dots, \alpha_n$ for which $\sum \alpha_s^2 = 1$, and $0 \le t \le T$, they have characteristic roots with negative real parts.

Theorem 1. If the coefficients of system (1) satisfy conditions I and II, then the normal fundamental matrix of solutions of the dual system (2) admits the estimate

$$\|Q(\alpha, t, t_0)\| \le C(1 + |\alpha|)^{(N-1)(p_1 - p_N)} \exp[-a|\alpha|^{p_N}(t - t_0)] \quad (a > 0), \tag{8}$$

so that, in particular, system (1) satisfies the condition (A) of I. G. Petrovskii.

The proof of Theorem 1 is obtained according to the following scheme. We rewrite system (3) in the form

$$\frac{dV}{dt} = A^0 V + (A - A^0)V \tag{9}$$

and denote by Q^* the normal fundamental matrix of solutions of the system

$$\frac{dV^*}{dt} = A^0 V^*, \tag{10}$$

an estimate which is easily obtained by applying the result from (1). Further, using the representation

$$Q(\alpha, t, t_0) = Q^*(\alpha, t, t_0) + \int_{t_0}^{t} Q^*(\alpha, t, \tau)\{A(\alpha, \tau) - A^0(\alpha, \tau)\} Q(\alpha, \tau, t_0)\, d\tau \tag{11}$$

and the known estimate of the matrix Q^*, we reduce the question of estimating the matrix Q to the question of estimating the solutions of a system of linear ordinary differential equations with constant coefficients.

From Theorem 1 and the known propositions from (1) there follows, in particular:

Theorem 2. *If the coefficients of system (1) satisfy conditions I and II, then for any initial functions* $\varphi(x) = \{\varphi_i(x_1, \dots, x_n)\}$*, continuous and bounded together with their partial derivatives up to some order* L*, in the strip* Ω $(0 \leq t \leq T;\ |x_s| < +\infty)$:

1. *There exists one and only one solution of the Cauchy problem* $u(x, t) = \{u_i(x_1, \dots, x_n; t)\}$*, belonging to the class of functions having continuous and bounded partial derivatives with respect to* x_s *up to order* p_1.
2. *If the initial functions* $\varphi(x)$ *and all their derivatives up to order* L *are sufficiently small, then the moduli of the functions* $u(x, t)$ *and of all their derivatives entering into (1) will also be small.*

When conditions I and II are fulfilled, the estimate (8) obtained by us makes it possible to obtain for system (1) the results established by I. M. Gel′fand and G. E. Shilov for systems parabolic in the sense of G. E. Shilov ((^{3}), p. 133). In particular, we can determine three characteristics: the reduced order p_0, the index of parabolicity h, and the genus μ of the matrix (4). We note only that, by virtue of estimate (8), for our systems the index of parabolicity is equal to p_N. According to the propositions of I. M. Gel′fand and G. E. Shilov ((^{3}), pp. 136 and 143), the class of correctness of the solution of the Cauchy problem for system (1) is the class of functions satisfying the inequality

$$|f(x)| \leq C \exp\left[b|x|^{\frac{p_0}{p_0-\mu}}\right], \qquad \text{when } \mu > 0;$$

$$|f(x)| \leq C_\varepsilon \exp\left[\varepsilon|x|^{\frac{p_N}{p_N-\mu}}\right] \text{ for any } \varepsilon > C, \qquad \text{when } \mu \leq 0,$$

and all solutions of the Cauchy problem for system (1) can be written with the aid of an integral of Poisson type in the form

$$u(x, t) = \int G(x - \xi, t, t_0)\varphi(\xi)\, d\xi,$$

where $G(x, t, t_0)$—the Green matrix—is the inverse Fourier transform of the matrix $Q(\alpha, t, t_0)$. Moreover, proceeding from estimate (8) and taking into account that p_N is an even number, we have proved for system (1) the following theorem:

Theorem 3. *If the coefficients of system (1) satisfy conditions I and II, then all solutions of the Cauchy problem belonging to the class of correctness are analytic in the variables* $x_1, \dots, x_n$ *for* $t > 0$.

Remark 1. In the particular case when in system (1) all weights p_j are equal to one another, condition I is fulfilled automatically, while fulfillment of condition II means that system (1) is parabolic in the sense of I. G. Petrovskii. Thus, the class of all systems whose coefficients satisfy conditions I and II extends the class of systems parabolic in the sense of I. G. Petrovskii. It is important to note that the class of systems of equations considered by us is such that the membership of a system in this class is determined only by the highest-order terms of the system (of course, by their own for each of the functions u_j). Such a property is not preserved for the class of all systems for which estimate (8) holds. We know that for every system parabolic in the sense of G. E. Shilov estimate (8) is necessarily fulfilled; however (see (⁴)), a change of the coefficient even of one lower-order term of such of the system can destroy its membership in the class, and we may obtain a system for which the Cauchy problem is posed incorrectly.

* It is sufficient, in any case, to take $L = (2n+1)[(p_1 - p_N)N + p_N]$.

Remark 2. Systems of the form (1) include more general systems of the form

$$\frac{\partial^{n_j} u_j}{\partial t^{n_j}} = \sum_{k=1}^{N} \sum_{(r_s)} a_{jk}^{(r_0 r_1 \dots r_n)}(t) \frac{\partial^{r_0 + r_1 + \dots + r_n} u_k}{\partial t^{r_0} \partial x_1^{r_1} \dots \partial x_n^{r_n}} \quad (r_0 < n_k;\ j = 1, 2, \dots, N). \tag{12}$$

The transition from system (12) to system (1) is carried out by introducing new unknown functions in place of the derivatives $\partial^{\nu} u_j / \partial t^{\nu}$ ($\nu = 1, 2, \dots, n_j - 1;\ j = 1, 2, \dots, N$).

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Note: Figure translations are in progress. See original paper for figures.