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Abstract

Full Text

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CONJUGATE NETS WITH PARABOLIC CONGRUENCES OF AXES

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MATHEMATICS

1. The line of intersection of the tangent planes of the first two Laplace transforms (A_1) and (A_2) of a conjugate net (A_0) is called the first axis, and the line A_1A_2, joining these transforms, is called the second axis of the net (A_0). The properties of hyperbolic congruences of axes of a conjugate net were first studied by Slotnik ([1]), and subsequently many other authors returned to them. Nets with parabolic congruences of axes were studied only by Mihăilescu ([2]).

In the present paper the study of nets of this kind is continued.

2. To each point of the conjugate net we attach a projective frame formed by four analytic points A_i: the point of the net A_0, the corresponding points A_1 and A_2 of the first Laplace transforms of the net (A_0), and an arbitrary point of the first axis A_3. The infinitesimal projective displacements of the frame are determined by the equations $dA_i = \omega_i^k A_k$ $(i, k = 0, 1, 2, 3)$. With the indicated choice of vertices,

$$\omega^3 = 0, \qquad \omega_1^3 = a\omega^1, \qquad \omega_2^3 = c\omega^2; \tag{1}$$

$$\omega_1^2 = 0, \qquad \omega_2^1 = 0. \tag{2}$$

Differentiating equations (2) with the aid of the structure equations of projective space $D\omega_i^k = [\omega_i^l \omega_l^k]$ and applying Cartan’ s lemma, we have

$$\omega_3^2 = a\omega^1 + \beta\omega^2, \qquad \omega_2^0 = \gamma_1\omega^1 - c\beta_1\omega^2,$$

$$\omega_1^0 = -a\beta\omega^1 + \gamma\omega^2, \qquad \omega_3^1 = \beta_1\omega^1 + \alpha_1\omega^2. \tag{3}$$

The invariant forms corresponding to the point and tangential Darboux invariants of the net (A_0) ([3]) are the forms

$$H = \gamma_1[\omega^1\omega^2], \qquad \bar{H} = c\alpha[\omega^1\omega^2],$$

$$K = \gamma[\omega^1\omega^2], \qquad \bar{K} = a\alpha_1[\omega^1\omega^2]. \tag{4}$$

3. The equations of the asymptotic lines of the surfaces (A_0), (A_1), (A_2) have the form

$$a(\omega^1)^2+c(\omega^2)^2 = 0, \qquad a\alpha(\omega^1)^2+\gamma(\omega^2)^2 = 0, \qquad \gamma_1(\omega^1)^2+c\alpha_1(\omega^2)^2 = 0. \tag{5}$$

The equalities

$$\gamma = c\alpha, \tag{6}$$

$$\gamma_1 = a\alpha_1 \tag{7}$$

are respectively the conditions that the congruences of tangents (A_0A_1) and (A_0A_2) be W-congruences. The tangent to the line $\omega^2 = 0$

on the surface (A_1) intersects the first axis at the point $T = A_3 - \beta A_0$, and the tangent to the line $\omega^1 = 0$ on the surface (A_2)–at the point $P = A_3 - \beta_1 A_0$. The points $F_{1,2} = \lambda A_0 + A_3$, $\Phi_{1,2} = \mu A_1 + A_2$, where λ and μ are the roots of the equations

$$\lambda^2 + (\beta + \beta_1)\lambda + \beta\beta_1 - \alpha\alpha_1 = 0, \tag{8}$$

$$a\gamma\mu^2 + ac(\beta - \beta_1) - c\gamma_1 = 0, \tag{9}$$

serve as the foci of the first and second axes. The first axis intersects the focal planes of the second, and the second–the focal planes of the first, respectively at the points $C_{1,2} = \bar{\mu}A_1 + A_2$, $B_{1,2} = \bar{\lambda}A_0 + A_3$, where $\bar{\lambda}$ and $\bar{\mu}$ are determined from the equations

$$ac\bar{\lambda}^2 + ac(\beta + \beta_1)\bar{\lambda} + ac\beta\beta_1 - \gamma\gamma_1 = 0, \tag{10}$$

$$\alpha\bar{\mu}^2 + (\beta - \beta_1)\bar{\mu} - \alpha_1 = 0. \tag{11}$$

Equations (8) and (10) show that the pairs of points (F_1, F_2) and (B_1, B_2) belong to the involution determined by the double point A_0 and the pair of points (T, P). When condition (6) is satisfied, from (9) and (11) we have that the pair of points (C_1, C_2) belongs to the involution determined by the double point A_1 and the pair of points (Φ_1, Φ_2).

4. A one-sided stratification of a pair of congruences of axes in the direction from the first axes to the second [4] is admitted by the nets R $(H = \bar{K},\ K = \bar{H})$. A particular solution with arbitrariness in 5 functions of one argument is formed by harmonic isothermally conjugate nets, i.e., such isothermally conjugate nets for which the points T and P coincide.

A one-sided stratification of a pair of congruences of axes in the direction from the second axes to the first is admitted by those and only those nets for which the sum of the point invariants is equal to the sum of the tangential invariants. This class of nets depends on one arbitrary function of two arguments. Examples of such nets are the nets R and the nets E $(H = \bar{H},\ K = \bar{K})$.

Finally, the nets R, and only these nets, admit a two-sided stratification of a pair of congruences of axes.

5. Nets for which the congruence of the first axes is parabolic are determined by the Pfaff equations (1), (2), (3) and the condition

$$(\beta - \beta_1)^2 + 4\alpha\alpha_1 = 0, \tag{12}$$

and nets with parabolic congruences of the second axes—by the same Pfaff equations and the condition

$$ac(\beta - \beta_1)^2 + 4\gamma\gamma_1 = 0. \tag{13}$$

Each of these classes of nets depends on one arbitrary function of two arguments. For nets of the first class the point $F = F_1 = F_2$ becomes the second double point of the involution considered on the first axis, and for nets of the second class the point $B = B_1 = B_2$ serves as the second double point of this involution.

Among the nets with parabolic congruences of the first axes with arbitrariness in 6 functions of one argument, there are distinguished the nets admitting a one-sided stratification of a pair of congruences of axes in the direction from the second axes to the first. An analogous result holds for nets with parabolic congruences of the second axes. The search for nets with parabolic congruences of either the first or the second axes that admit a one-sided stratification from the first axes to the second leads to the nets R, for which both congruences of axes are parabolic. With arbitrariness in 6 functions of one argument there exist nets with parabolic congruences of either the first or the second axes, characterized by condition (6). For the per-

ones the point $C = C_1 = C_2$, while for the second ones the point $\Phi = \Phi_1 = \Phi_2$ becomes the second double point of the involution considered on the second axis.

6. The system of equations (1), (2), (3) under conditions (12), (13) determines, with arbitrariness in 6 functions of one argument, nets for which both axis congruences are parabolic. Let us bring the vertex A_3 together

with the focus F. Then $\beta_1 = -\beta$; $A_3 = F = B$; $\Phi = -\frac{c\beta}{\gamma} A_1 + A_2$; $C = -\frac{\beta}{\alpha} A_1 + A_2$. It is easy to see that the cross ratio of the points A_1, A_2, Φ, C is equal to the ratio of the first tangential invariant to the second point invariant. Conditions (12) and (13) give

$$\gamma\gamma_1 = ac\alpha\alpha_1, \tag{14}$$

i.e. the product of the Darboux tangential invariants is equal to the product of the point invariants. Condition (14), in view of (5), also means that on the surfaces (A_1) and (A_2) there correspond asymptotic lines ([2]).

The asymptotic lines on the surfaces (A_3) and (Φ), whose tangents describe the congruences of the first and second axes of the net (A_0), are determined respectively by the equations

$$\beta\omega^1 - \alpha_1\omega^2 = 0, \tag{15}$$

$$a\beta\omega^1 - \gamma\omega^2 = 0, \tag{15'}$$

or, in view of (12) and (13), by the equations

$$\alpha\omega^1 + \beta\omega^2 = 0, \tag{16}$$

$$\gamma_1\omega^1 + c\beta\omega^2 = 0. \tag{16'}$$

The geometric characteristic of the nets (15) and (16) is as follows: the tangents to the lines of the second families of these nets, respectively on the surfaces (A_1) and (A_2), pass through the point A_3; the cross ratios of two directions tangent to the lines of the focal net on the surface (A_1), with the two directions tangent to the lines of the nets (15), (15′) and (16), (16′), are respectively equal to the ratios of the second or first point and tangential Darboux invariants of the net (A_0). The conjugacy of the directions (15), (15′) and (16), (16′) on the surface (A_1) is a necessary and sufficient condition for, respectively, the congruence of the first axes and the congruence of the second axes to be parabolic.

7. Nets R with parabolic axis congruences are determined by equations (1), (2), (3), (6), (7), (12). Investigation of this system shows that these are either Wilczynski nets, or nets for which $H = \bar{H} = K = \bar{K}$ ([2]).

The even Laplace transforms (M_{2k}) of Wilczynski nets with parabolic axis congruences are situated on the first axis of the net, the odd ones (M_{2k-1})—on the second; taking, as in item 6, $F = A_3,\ \beta_1 = -\beta$ and carrying out the corresponding normalization, we obtain

$$M_{2n} = nA_3 - \beta A_0, \qquad M_{2n-1} = -n\beta A_1 + (n-1)\alpha A_2,$$

$$M_{-2n} = nA_3 + \beta A_0, \qquad M_{-2n+1} = (1-n)\beta A_1 + n\alpha A_2 \quad (n = 0, 1, 2, \ldots). \tag{17}$$

It is easy to see that

$$(A_0, A_3, M_{2n} M_{-2n}) = -1 \quad (n \geqslant 1); \qquad (A_1 A_2, M_{2n-1} M_{-2n+1}) = \frac{n^2}{(n-1)^2};$$

$$(\Phi A_1, M_{2n+1} M_{-2n+1}) = -1 \quad (n > 1); \qquad (\Phi A_2, M_{2n-1} M_{-2n-1}) = -1 \quad (n > 1). \tag{18}$$

The congruences of the tangent focal nets of this sequence belong to the linear complexes of one pencil, determined by the complexes

$$p_{02} - \alpha p_{13} - \beta p_{23} = 0, \qquad p_{01} + \beta p_{13} - \alpha_1 p_{23} = 0. \tag{19}$$

Two special complexes of this pencil coincide. Both congruences of axes belong to all complexes of this pencil and therefore coincide, forming a linear congruence belonging to the unique special complex of the pencil. The pair of directrices of it coincides with the line $A_3\Phi$. The rays of this linear congruence split into two subfamilies: the rays of one describe the first axes of the net (A_0), and the rays of the other—the second. Since the net (A_0) is an R-net, it follows from item 4 that these two subfamilies of rays are doubly developable. The developable surfaces of the first family are cones with vertices at A_3 and with directrices (15) on the surface (A_0), while those of the second are cones with vertices at Φ and with directrices (15′) on the surface (A_0). The directions (15) and (15′) are conjugate on (A_0). The tangents to the lines (15) describe the congruence $(A_0\Phi)$. The sequence of points $\{M_{2k}\}$ converges to the point A_3, and the sequence of points $\{M_{2k-1}\}$ to the point Φ.

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Note: Figure translations are in progress. See original paper for figures.