MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE EXISTENCE OF BOUNDED SOLUTIONS FOR SOME NONLINEAR DIFFERENTIAL SYSTEMS

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Let $a(t)$ be a real continuous function defined on the half-axis $t \geqslant 0$.

According to Perron (1), a function $a(t)$ satisfies condition (A) if the functions $\exp\left\{\int_0^t a(\tau)\,d\tau\right\}$ and $\int_0^t \exp\left\{\int_\tau^t a(\theta)\,d\theta\right\}\,d\tau$ are bounded on the half-axis $t \geqslant 0$.

Similarly, a function $a(t)$ satisfies condition (B) if the function $\exp\left\{\int_0^t a(\tau)\,d\tau\right\}$ is unbounded, while $\int_t^\infty \exp\left\{\int_\tau^t a(\theta)\,d\theta\right\}\,d\tau$ is defined and bounded on the half-axis $t \geqslant 0$.

In what follows we shall consider only real functions. Consider the system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_i) + R_i(t, x_1, \dots, x_n), \qquad 1 \leqslant i \leqslant n, \tag{1}$$

assuming that the following conditions are fulfilled:

a) The functions $f_i(t, x_1, \dots, x_i)$, $1 \leqslant i \leqslant n$, are continuous on the set

$$t \geqslant 0, \qquad \|x\| \leqslant h, \qquad 0 < h \leqslant +\infty, \tag{$\bar{\Delta}$}$$

and $f_i(t, 0, \dots, 0) \equiv 0, \quad 1 \leqslant i \leqslant n$.

b) The derivatives $\partial f_i/\partial x_j$, $1 \leqslant j \leqslant i \leqslant n$, are continuous and bounded in $\bar{\Delta}$.

c) For each index i, $1 \leqslant i \leqslant n$, there exists a function $a_i(t)$, continuous on the half-axis $t \geqslant 0$, satisfying condition (A) or (B), such that $\partial f_i/\partial x_i \leqslant a_i(t)$ in the case where $a_i(t)$ satisfies condition (A), and $\partial f_i/\partial x_i \geqslant a_i(t)$ if $a_i(t)$ satisfies condition (B). Obviously, the preceding inequalities hold throughout $\bar{\Delta}$.

In what follows we shall denote by i_A those indices i, $1 \leqslant i \leqslant n$, for which $a_i(t)$ satisfies condition (A).

d) The functions $R_i(t, x_1, \dots, x_n)$, $1 \leqslant i \leqslant n$, are continuous in $\bar{\Delta}$ and satisfy on this set the Lipschitz condition

$$|R_i(t, x_1, \dots, x_n) - R_i(t, \bar{x}_1, \dots, \bar{x}_n)| \leqslant L_i \|x - \bar{x}\|, \qquad 1 \leqslant i \leqslant n. \tag{2}$$

Theorem 1. Let the functions f_i and R_i satisfy conditions a), b), c), and d).

Then system (1), under the conditions

$$x_i(0) = x_i^{(0)}, \qquad i = i_A, \tag{3}$$

admits a unique solution, defined on the half-axis $t \geq 0$, whose graph belongs to $\overline{\Delta}$, provided that $x_i^{(0)}$, $i = i_A$, and L are sufficiently small.

If the indices i_A are absent, then system (1) has a unique solution, defined on the half-axis $t \geq 0$, whose graph belongs to $\overline{\Delta}$.

The proof of this theorem is based on the Banach fixed-point principle and on certain lemmas concerning systems of the form

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_i) + h_i(t), \qquad 1 \leq i \leq n, \tag{4}$$

where the f_i satisfy the conditions indicated above, and the $h_i(t)$ are continuous bounded functions on the half-axis $t \geq 0$. The proof also implies the uniform convergence (on the half-axis $t \geq 0$) of the process of successive approximations defined by the relations

$$\frac{dx_i^{(k)}}{dt} = f_i(t, x_1^{(k)}, \dots, x_i^{(k)}) + R_i(t, x_1^{(k-1)}, \dots, x_n^{(k-1)}), \qquad 1 \leq i \leq n;\ k \geq 1; \tag{5}$$

$$x_i^{(k)}(0) = x_i^{(0)}, \qquad i = i_A,\ k \geq 1. \tag{6}$$

Theorem 1 generalizes Perron' s result ([1]), since any linear differential system can be reduced to triangular form. Restricting ourselves to the class of real functions leads to no further simplifications, except in notation.

Let us note that an analogous theorem is also valid in the case where solutions bounded on the entire real axis are sought. This theorem is easily formulated by taking into account the results of ([2a]).

We replace conditions b) and c) by the more general conditions:

b$_1$) The derivatives $\partial f_i / \partial x_i$, $1 \leq i \leq n$, are continuous and bounded in $\overline{\Delta}$, and

$$|f_i(t, x_1, \dots, x_{i-1}, 0)| \leq M \sum_{j=1}^{i-1} |x_j|, \qquad 2 \leq i \leq n, \tag{7}$$

where M is a positive constant.

c$_1$) The functions $R_i(t, x_1, \dots, x_n)$, $1 \leq i \leq n$, are continuous in $\overline{\Delta}$.

Theorem 2. Suppose that the functions f_i and R_i satisfy conditions a), b$_1$), c), and c$_1$).

Then system (1), under conditions (3), has at least one solution defined on the half-axis $t \geq 0$, whose graph belongs to $\overline{\Delta}$, provided that $x_i^{(0)}$, $i = i_A$, and R_i are sufficiently small.

In the case where the indices i_A are absent, system (1) has at least one solution defined on the half-axis $t \geq 0$, whose graph belongs to $\overline{\Delta}$.

For the proof of this theorem one uses, as was indicated in (^{2b}), the Schauder-Tikhonov fixed-point principle and the following lemma:

Lemma. Suppose that the functions f_i satisfy conditions a), b$_1$), and c), and that the $h_i(t)$ are continuous and bounded on the half-axis $t \geq 0$.

Then system (4), under conditions (3), has a unique solution, defined on the half-axis $t \geq 0$, whose graph belongs to $\overline{\Delta}$, provided that $x_i^{(0)}$, $i = i_A$, and $h_i(t)$ are sufficiently small.

For this solution the inequality holds

$$\|x(t)\| \leq P \sum_{j=i_A} |x_j^{(0)}| + Q\|h(t)\|, \tag{8}$$

where P and Q are positive quantities depending only on the functions f_i.

When the indices i_A are absent, system (4) has a unique solution, defined on the half-axis $t \geq 0$, whose graph belongs to $\overline{\Delta}$. Inequality (8) holds in this case if one sets $P = 0$.

By $\|x(t)\|$ we denote $\max_i \left\{\sup_{t\geq 0} |x_i(t)|\right\}$, $1 \leq i \leq n$.

This lemma is also useful for proving the fact that the zero solution of the unperturbed system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n), \qquad 1 \leq i \leq n, \tag{1'}$$

is conditionally stable under constantly acting perturbations (2).

Let us now consider the system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n), \qquad 1 \le i \le n, \tag{9}$$

and suppose that the following conditions are satisfied:

a′) The functions f_i are continuous in the set $\overline{\Delta}$.

b′) The derivatives $\partial f_i / \partial x_i$, $1 \le i \le n$, are continuous in $\overline{\Delta}$.

c′) The functions $f_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$, $1 \le i \le n$, are sufficiently small in $\overline{\Delta}$.

Theorem 3. Let the functions f_i satisfy conditions a′), b′), c) and c′).

Then system (9), under conditions (3), admits at least one solution, defined on the half-axis $t \ge 0$, whose graph belongs to $\overline{\Delta}$.

This theorem generalizes some previously obtained results (2). One can obtain an analogous theorem on the existence of a family of bounded solutions on the entire real axis.

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Note: Figure translations are in progress. See original paper for figures.