Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1960. Volume 130, No. 2

PHYSICS

Yu. A. Tserkovnikov

On the Question of the Convective Instability of Plasma

(Presented by Academician N. N. Bogolyubov, 16 VII 1959)

In paper (1), the convective instability of plasma, i.e., instability arising as a consequence of spatial inhomogeneity of the parameters characterizing it, was considered on the basis of kinetic equations without taking into account collisions between particles (2). Such an approach is quite satisfactory in the study of a rarefied plasma; however, the equations obtained in this way are very complicated. In the present work we shall consider the same problem as in (1), proceeding from the simpler hydrodynamic equations. In solving the equations we shall restrict ourselves to studying the behavior of small-scale perturbations, without taking boundary conditions into account, which is justified in the investigation of convective instability.

Let us first consider, as an example, the simple case of an infinitely conducting fluid described by the equations of magnetic hydrodynamics

$$\rho\left\{\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}\right\}=-\nabla\left(p+\frac{H^2}{8\pi}\right)+\rho\mathbf{g},$$

$$\frac{\partial \rho}{\partial t}+(\mathbf{u}\nabla)\rho+\rho\operatorname{div}\mathbf{u}=0,\qquad \frac{\partial p}{\partial t}+(\mathbf{u}\nabla)p+\gamma p\operatorname{div}\mathbf{u}=0,\tag{1}$$

$$\frac{\partial \mathbf{H}}{\partial t}=\operatorname{rot}[\mathbf{u}\mathbf{H}]=-(\mathbf{u}\nabla)\mathbf{H}-\mathbf{H}\operatorname{div}\mathbf{u}.$$

Everywhere we shall assume that the fluid is homogeneous along the magnetic field $\mathbf{H}$, $\mathbf{g}\perp\mathbf{H}$ ($\mathbf{g}$ is the acceleration due to gravity), and the magnetic field is not curved. We shall investigate stability in the linear approximation, considering the perturbations to be small in comparison with the stationary values of the quantities, for example, $p=p_0+p'$, $p'\ll p_0$. If the solution of the linearized system of equations grows without bound, the plasma is regarded as unstable.

Consider such a stationary state in which $\mathbf{u} = 0$ and, consequently,

$$\nabla \left(p_0 + \frac{H_0^2}{8\pi} \right) = \rho_0 \mathbf{g}. \quad (2)$$

The equations for the perturbations, if one introduces the displacement vector $\vec{\xi}$ such that $\mathbf{u} = \partial\vec{\xi}/\partial t$, then reduce to the equation

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \nabla \left\{ (\vec{\xi}\nabla) \left(p_0 + \frac{H_0^2}{8\pi} \right) + \gamma \left(p_0 + \frac{H_0^2}{4\pi\gamma} \right) \operatorname{div} \vec{\xi} \right\} - \mathbf{g} \left\{ (\vec{\xi}\nabla)\rho_0 + \rho_0 \operatorname{div} \vec{\xi} \right\}. \quad (3)$$

We shall now assume that the length λ on which the perturbations change appreciably is much smaller than the length L on which the stationary quantities change appreciably ($\lambda \ll L$). Then the gradients of the stationary quantities may be considered small, proportional to the small parameter $\mu = \lambda/L$.

If in equation (3) the gradients of the stationary quantities are neglected (by virtue of (2), $\mathbf{g} \sim \mu$), we obtain an equation describing the propagation

sound waves in a direction perpendicular to the magnetic field, with velocity $c_{\text{sv}} = \sqrt{\gamma(p_0 + H_0^2/4\pi\gamma)/\rho_0}$. Let us now consider slow processes, assuming that $\partial/\partial t \sim \mu$. Discarding in (3) quantities $\sim \mu$, we obtain $\operatorname{div} \vec{\xi} \sim \mu \simeq 0$. In the next approximation we find

$$\operatorname{div} \vec{\xi} = -\frac{(\vec{\xi}\nabla)(p_0 + H_0^2/8\pi)}{\gamma(p_0 + H_0^2/4\pi\gamma)}.$$

Now applying the operation rot twice to both sides of equation (3), and using the fact that the motion is approximately vortical, $\operatorname{div} \vec{\xi} \sim \mu$, for the quantity $\chi = (\mathbf{g}\vec{\xi})$ we shall have the equation

$$-\frac{\partial^2}{\partial t^2} \Delta\chi = \frac{\rho_0 Q}{p_0 + H_0^2/8\pi} \{ g^2 \Delta\chi - (\mathbf{g}\nabla)^2 \chi \}, \quad (4)$$

where $Q = \partial \ln \rho_0 / \partial \ln(p_0 + H_0^2/8\pi) - (p_0 + H_0^2/8\pi)/\gamma(p_0 + H_0^2/4\pi\gamma)$. The stationary quantities in deriving (4) we did not differentiate, since this would correspond to taking into account terms $\sim \mu^3$.

It remains to solve equation (4). We shall seek a solution in the form $\chi = a \exp(iS + i\omega t)$, where a, ω, and $\mathbf{k} = \nabla S$ are slowly varying functions over the wavelength determined by the wave vector $\mathbf{k}$ ($|k| \sim 1/\lambda$). If $\operatorname{Im} \omega < 0$, the superposition of such solutions determines, as $t \to \infty$, the evolution of an initial small-scale perturbation $\chi_0 = \sum_n a_n \exp iS_n + \text{const}$, expanded in a complete orthonormal system of functions $\exp iS_n$. In this case the frequency ω is a pole of the Laplace transform of the function χ. From (4) we obtain

$$\omega^2 = ((k^2 - k_g^2)/k^2)\,(\rho_0 g^2/(p_0 + H_0^2/8\pi))\,Q, \tag{5}$$

where k_g is the component of $\mathbf{k}$ perpendicular to $\mathbf{g}$.

Thus, the condition $Q > 0$ is the condition of convective stability for the system described by equations (1). We note that the stability condition does not include the vector $\mathbf{k}$, which characterizes the shape and dimensions of the perturbation. For $H_0 = 0$ this condition becomes the usual condition for convective stability of a gas of uncharged particles (see ([3]), p. 22).

It follows from (5) that convection is due to the force of gravity, since for $g = 0$, $\omega^2 = 0$. However, cases may occur (gas discharge) when the force of gravity can be neglected and the equilibrium condition has the form

$$\nabla(p_0 + H_0^2/8\pi) = 0. \tag{6}$$

Let us consider the more complicated case of a two-component system in the adiabatic approximation (without taking heat fluxes into account) and show, by the same method as above, that such a system will be unstable even in the absence of the force of gravity. Under the previous assumptions for perturbations we obtain the system of equations

$$Mn_0\frac{\partial \mathbf{u}}{\partial t} + mn_0\frac{d\mathbf{v}}{dt} = -\nabla\left(p + \frac{H_0 H'}{4\pi}\right) + e(n_i' - n_e')\mathbf{E}_0,$$

$$Mn_0\frac{\partial \mathbf{u}}{\partial t} = -\nabla p_i' + \nabla p_{i0}\frac{n_i'}{n_0} + en_0\left(\mathbf{E} + \frac{1}{c}[\mathbf{u}\mathbf{H}_0]\right),$$

$$\frac{\partial n_i'}{\partial t} + (\mathbf{u}\nabla)n_0 + n_0\,\mathrm{div}\,\mathbf{u} = 0, \qquad \frac{\partial p_i'}{\partial t} + (\mathbf{u}\nabla)p_{i0} + \gamma p_{i0}\,\mathrm{div}\,\mathbf{u} = 0,$$

$$\frac{dn_e'}{dt} + (\mathbf{v}\nabla)n_0 + n_0\,\mathrm{div}\,\mathbf{v} = 0, \qquad \frac{dp_e'}{dt} + (\mathbf{v}\nabla)p_{e0} + \gamma p_{e0}\,\mathrm{div}\,\mathbf{v} = 0, \tag{7}$$

where $\mathbf{u}$ is the ion velocity; $\mathbf{v}$ is the electron velocity; $p = p_i' + p_e'$; $p_{i0} = p_{e0} = p_0/2$; $d/dt = \partial/\partial t + (\mathbf{v}_0\nabla)$; $\mathbf{v}_0 = -(c/en_0H_0^2)[\mathbf{H}_0\nabla p_0]$ is the velocity of the electrons in the stationary state.

The fields $\mathbf{E}$ and $\mathbf{H}$ are determined by Maxwell' s equations

$$\frac{\partial \mathbf{H}}{\partial t} = -c\ \mathrm{rot}\,\mathbf{E}', \qquad \mathrm{div}\,\mathbf{E}' = 4\pi e(n_i' - n_e'). \tag{8}$$

We shall solve the system of equations (7), (8) by the method of expansion in the small parameter μ, assuming $\partial/\partial t$ and $d/dt \sim \mu$. Neglecting, in equations (7), the gradients of stationary quantities, we obtain the relations

$$\nabla\left(p + \frac{(\mathbf{H}_0\mathbf{H}')}{4\pi}\right) \simeq 0, \qquad -\nabla p_i' + en_0\left(\mathbf{E}' + \frac{1}{c}[\mathbf{u}\mathbf{H}_0]\right) \simeq 0, \operatorname{div}\mathbf{u} \simeq \operatorname{div}\mathbf{v} \simeq 0. \tag{9}$$

With the aid of (8), the second of equations (9) can be written in the form

$$n_i' - n_e' = \Delta p_i'/4\pi e^2 n_0 - (\mathbf{H}_0 \operatorname{rot}\mathbf{u})/4\pi ec. \tag{10}$$

Let us estimate the order of the terms standing on the right-hand side of this equation: $\Delta p_i'/4\pi e^2 n_0 \sim k^2 p_i/4\pi e^2 n_0 \sim (T_0 k^2/4\pi e^2 n_0) n_i \sim (kd)^2 n_i$, where $d = T_0/4\pi e^2 n_0$ is the Debye length, T_0 is the temperature in energy units. An analogous estimate holds for the second term. We shall assume that the wavelength characterizing the dimensions of the perturbation is much greater than d ($kd \ll 1$). In this approximation the plasma may be regarded as quasineutral, and instead of (9) we shall have

$$n_e' \simeq n_i' = n, \qquad p \simeq -(\mathbf{H}_0\mathbf{H}')/4\pi, \qquad \mathbf{u} - \mathbf{v} \simeq -(c/4\pi e n_0)\operatorname{rot}\mathbf{H}'. \tag{11}$$

Using relations (11), and substituting $\mathbf{E}$ from the second equation (7) into the first equation (8), it is not difficult to reduce equations (7)–(8) to the system

$$\frac{\partial\chi}{\partial t} + \frac{m}{M}\frac{d\chi}{dt} = \frac{m}{M}(\mathbf{v}_0\nabla)\frac{dp}{dt}, \qquad (1+\kappa)\frac{d}{dt}\frac{\partial\eta}{\partial t} + \frac{Q}{2}\left(1 - \frac{Q}{2}\right)(\mathbf{v}_0\nabla)\eta = 0, \tag{12}$$

where $\chi = (\mathbf{u}\nabla p_0)$, $\eta = \partial p/\partial t + \chi$, $Q = \partial \ln p_0 n_0^{-\gamma}/\partial \ln p_0$, and $\kappa = 4\pi\gamma p_0/H_0^2$.

The system of equations (12) describes motions of two types. Indeed, suppose, as is often the case, that the mass of the negative ions is much less than the mass of the positive ones ($m/M \ll 1$). Then, dropping in the first equation the terms $\sim m/M$, we obtain $\chi \simeq 0$, $\eta \simeq \partial p/\partial t$. The equation for the frequency ω, obtained from the second equation (12), has a solution with imaginary part equal to zero. Thus, this root does not lead to instability.

Let us now suppose that $\partial/\partial t \sim \sqrt{m/M}$; then from equations (12) we shall have $\eta \simeq 0$, $\chi \simeq -\partial p/\partial t$, and

$$-\frac{\partial^2 p}{\partial t^2} = \frac{m}{M}(\mathbf{v}_0\nabla)^2 p. \tag{13}$$

Solving this equation, we obtain

$$\omega^2 = -\frac{m}{M}(\mathbf{k}\mathbf{v}_0)^2, \tag{14}$$

i.e., the system described by equations (7) will be unstable if only in the stationary state there is a current $\mathbf{j}_0 = -en_0\mathbf{v}_0$.

Finally, let us take into account the influence of heat fluxes on processes of the convective type. The hydrodynamic equations for a two-component plasma were obtained in work ([4]). Assuming that the Larmor frequency $\omega_H = eH/mc$ is much greater than the collision frequency, in the first nonvanishing approxi-

···for the heat flux $\mathbf{q}$ (the terms $\operatorname{div}\mathbf{q}'$ will additionally enter the equations for p' in (7)) we obtain from the work formula (4)

$$\mathbf{q} = \gamma\,\frac{cp}{eH^2}\left[\mathbf{H}\nabla\frac{p}{n}\right], \tag{15}$$

where $\gamma = 5/3$. We shall omit, where possible, the index distinguishing the plasma components. If one starts from the Boltzmann equation without taking collisions into account, solving it by expansion in powers of $1/H$ (5-7), and takes the Maxwellian distribution as the zeroth approximation, then for $\mathbf{q}$ one also obtains expression (15), in which $\gamma = 2$ and $p \to p_\perp$, where $p_\perp$—the component of the pressure tensor $\mathbf{P}$ perpendicular to the magnetic field—is determined by the equation

$$\frac{\partial p_\perp}{\partial t} + (\mathbf{u}\nabla)p_\perp + \gamma p_\perp \operatorname{div}\mathbf{u} + \operatorname{div}\mathbf{q} = 0. \tag{16}$$

For the velocity $\mathbf{u}$, the equation is

$$mn\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u}\right) = -\operatorname{div}\mathbf{P} + en_0\left(\mathbf{E} + \frac{1}{c}[\mathbf{u}\mathbf{H}]\right). \tag{17}$$

The off-diagonal elements of the pressure tensor will be $\sim 1/H^2$. Discarding terms $\sim 1/H^2$, from (17) we shall have

$$\mathbf{u} = (c/enH^2)[\mathbf{H}\nabla p_\perp] + (c/H^2)[\mathbf{E}\mathbf{H}]. \tag{18}$$

Now we can linearize equations (15)—(18) and solve them by means of an expansion in the small parameter $\mu = \lambda/L$. Discarding terms $\sim \mu$, we again arrive at relations (11) ($p \to p_\perp$). Using them, instead of the second equation (12) we shall have (the notation is the same)

$$\frac{\partial}{\partial t}\frac{\partial \eta}{\partial t} + \frac{1}{4}\left[1 - \frac{\varkappa^2}{\gamma} + \frac{2\varkappa^2 Q}{\gamma(1+\varkappa)}\right](\mathbf{v}_0\nabla)^2\eta = 0, \tag{19}$$

whence for the quantity ω we obtain

$$\omega = \frac{1}{2}(\mathbf{kv}_0)\left\{-1 \pm \sqrt{(\varkappa/\gamma)\left(1 - \frac{2Q}{1+\varkappa}\right)}\right\}. \tag{20}$$

Thus, owing to the drift heat fluxes (15), an instability is possible that was absent in the preceding case. The stability condition obviously has the form ($Q = 1 - \gamma + \gamma\,\partial \ln T_0/\partial \ln p_0$)

$$\frac{\partial \ln T_0}{\partial \ln p_0} < 1 + \frac{(\varkappa - 1)}{2\gamma}. \tag{21}$$

If, taking into account terms $\sim 1/H^2$, it turns out that $\mathrm{rot}\,\mathrm{rot}\,\mathrm{div}\,\mathbf{P} = 0$, then the roots (14) will also be obtained, since in this case ($\partial/\partial t \sim \sqrt{m/M}$) and from (19) again $\chi = -\partial p/\partial t$.

A shortcoming of the approach set forth here to the investigation of stability, if the plasma is highly rarefied, is that in obtaining relation (15) the Maxwellian distribution was taken as the zeroth approximation in $1/H$. More rigorously, on the basis of kinetic equations without allowance for collisions, the convective stability of a rarefied plasma was studied in work (1).

In conclusion I express my deep gratitude to N. N. Bogolyubov for his attention to the work.

Received
22 VI 1959

CITED LITERATURE

1. Yu. A. Tserkovnikov, ZhETF, **32**, 67 (1957).
2. A. A. Vlasov, ZhETF, **8**, 291 (1938).
3. L. D. Landau, E. M. Lifshitz, *Mechanics of Continuous Media*, Moscow, 1953.
4. S. I. Braginskii, ZhETF, **33**, 459 (1957).
5. G. Chew, M. Goldberger, F. Low, Proc. Roy. Soc., A **236**, 112 (1956).
6. K. M. Watson, Phys. Rev., **102**, 12 (1956).
7. K. A. Brueckner, K. M. Watson, Phys. Rev., **102**, 19 (1956).

Note: Figure translations are in progress. See original paper for figures.