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Abstract

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ON THE RI^*-SOLVABLE RADICAL OF GROUPS

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The idea of constructing the RI^*-solvable radical of groups and the concepts necessary for carrying it out are drawn from the theory of associative rings (^{1,2}), and their transfer to groups is effected by replacing multiplication in rings by the operation of commutation in groups. We note that this same idea is also used in (^{3}), where, however, only groups satisfying the maximal condition for normal divisors are considered*.

§ 1. Let us first note that everywhere below $[A, B]$ denotes the mutual commutant of certain subsets A and B of a group G. By analogy with the corresponding concept in associative rings (^{1}), we give the following

Definition 1. A normal divisor P of a group G, distinct from G, will be called a **prime normal divisor** if from the relation $[A, B] \subseteq P$, where A and B are invariant subgroups of the group G, there follows at least one of the inclusions $A \subseteq P$, or $B \subseteq P$.

In (^{3}) this concept is considered and studied in groups satisfying the maximal condition for normal divisors. In the general case one can indicate a number of characteristics of a prime normal divisor (^{4}), but for our purposes the following proved to be the most effective:

A normal divisor P of a group G, $P \neq G$, is a prime normal divisor of this group if and only if from the relation $[[G, a], b] \subseteq P$, where a, b are elements of G, it follows that either $a \in P$ or $b \in P$.

It is clear that every prime normal divisor P of a group G always contains: a) all solvable invariant subgroups of the group G; b) the centralizer of any normal divisor A of the group G, provided that A itself is not contained in P. In resolving the question of the existence in a group of prime normal divisors, the following concept plays an important role (cf. (^{2})).

Definition 2. A sequence $x_0, x_1, \ldots$ of elements of a group G will be called a k-sequence if in G there exist elements $y_0, y_1, \ldots$ such that

$$x_{i+1} = [[y_i, x_i], x_i]$$

for all $i = 0, 1, \ldots$.

We shall call the k-sequence $x_0, x_1, \ldots$ terminating if for some n one has $x_n = 1$, and nonterminating otherwise.

Theorem 1. *A group G contains a prime normal divisor if and only if in G there exists a nonterminating k-sequence $x_0, x_1, \ldots$ of elements of this group.*

In the proof of this theorem, together with both definitions of a prime normal divisor, the principle of the maximal element is used.

As usual, by an RI^***-solvable group** we shall mean a group which possesses a solvable ascending invariant series. From characterization of this class of groups given in (5), and Theorem 1, immediately yields

Theorem 2. *A group G contains no prime normal divisors if and only if it is an RI^*-solvable group.*

In what follows we shall use the following very evident consequence of the theorems just stated.

Corollary. *If a subgroup A of a group G is not RI^*-solvable, then G has a prime normal divisor P such that A is not contained in P.*

§ 2. We now turn to the consideration of the class of groups which corresponds here to prime rings (1).

Definition 3. A group G will be called a **prime group** if its identity normal divisor is a prime normal divisor.

From property a) of a prime normal divisor, noted in § 1, it is quite clear that a group G is a prime group if and only if every nonidentity invariant subgroup of it has in the whole group a centralizer coinciding with the identity. In this sense a prime group is a "group without centralizers." Prime groups include the simple noncommutative groups, the symmetric groups S_n for $n > 4$, and also free noncommutative groups.

There is a deeper connection between prime groups and prime normal divisors, as the following shows.

Lemma 1. *A normal divisor P of a group G is a prime normal divisor if and only if the factor group G/P by it is prime.*

Together with the results of the preceding paragraph, this lemma makes it possible to draw the following conclusions:

1. *If $\bar{G}$ is a homomorphic image of a group G, then a normal divisor $\bar{P}$ is a prime normal divisor of the group $\bar{G}$ if and only if its complete inverse image P in G is a prime normal divisor.*

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[*] The principal results of the present note were reported by the author at the Second All-Union Colloquium on General Algebra, and some are contained in (4).

2. *For RI^*-solvability of a group G it is necessary and sufficient that G not be mapped homomorphically onto prime groups.*
3. *A group G contains in every homomorphic image $\bar{G} \neq 1$ a nonidentity normal divisor with centralizer distinct from the identity if and only if G is an RI^*-group.*

Of interest are certain special classes of prime groups. As usual, let us call a group G **directly indecomposable** if in every isomorphic representation of it as a direct product there is a factor isomorphic to G. As is known, this is equivalent to the assertion that the intersection of all nonidentity normal divisors of the group G is itself distinct from the identity, i.e. is the unique minimal normal divisor of the group G. We shall call it the **core** of the group G and denote it by C. Since $[C, C] \subset C$, by the minimality of C, always either $[C, C] = 1$ or $[C, C] = C$. In the first case C is abelian, and in the second we shall call it an **idempotent core**. As for rings (6), the following fact is easily proved:

A group G is directly indecomposable with idempotent core if and only if it is prime and contains a minimal normal divisor.

It is obvious that, apart from the free noncommutative groups, the prime groups from the examples given above are directly indecomposable. Of special interest are the prime directly indecomposable groups whose core is completely reducible. They belong to the class of the so-called F_1-semisimple groups and admit a description

(7). The existence in a primary subgroup of an indecomposable group G of a simple noncommutative subinvariant subgroup is a necessary and sufficient condition for the complete reducibility of the core C of this group, which follows from (8). However, whether in the general case the core C will be completely reducible or not remains unclear.

Finally, let us note that, using (9), it is easy to obtain the following result:

A primary group possessing a nonidentity finite class of conjugate elements is a finite group.

It is clear from this that *primary locally normal groups, primary groups with finite classes, in particular layer-finite primary groups, are finite*, or, in view of Lemma 1, this means that *in the indicated classes of groups primary normal divisors have finite index.*

§ 3. Using the notions of the preceding paragraphs, we shall now construct in groups a radical similar to the Baer-McCoy radical in associative rings (1,2).

Definition 4. By the **radical** R of a group G we shall mean the intersection of all primary normal divisors of this group.

We easily obtain, relying on the corollary of § 1, that R is a characteristic RI^*-subgroup of the group G. In a finite group the radical R coincides with the

Fitting soluble radical, while in the general case the following properties of it hold.

Theorem 3. *The radical R of a group G consists of those and only those elements x of the group G for which every k-sequence $x = x_0, x_1, \ldots$ of this group generated by x is terminating.*

Theorem 4. *If R is the radical of the group G, then the radical of the factor group G/R coincides with the identity subgroup.*

Just as in the case of associative rings (2), one could arrive at the definition of the radical R by another route. To this end, denote by $A(G)$ the product of all abelian normal divisors of the group G, and construct in G an ascending series of characteristic subgroups

$$1 = A_0(G) \subset A_1(G) \subset \cdots \subset A_\alpha(G) \subset A_{\alpha+1} \subset \cdots,$$

where $A_{\alpha+1}(G)/A_\alpha(G) = A(G/A_\alpha(G))$, and at limit places, as usual, the union of all preceding terms is taken. Let λ be the first ordinal number such that $A_\lambda(G) = A_{\lambda+1}(G)$. We shall call $A_\lambda(G) = \widetilde{R}$ the **upper radical** of the group G.

Theorem 5. *The radical R of the group G coincides with the upper radical $\widetilde{R}$ of this group.*

The proof of this result, apart from Theorem 4, essentially relies on the following lemma:

Lemma 2. *Let N be such a normal divisor of a group G that the factor group G/N contains no nonidentity abelian normal divisors, and let $N \subset M$, where M is a normal divisor of the group G. Then in the group G there exists a primary normal divisor P with the condition $P \supset N$ and $P \not\supset M$.*

As is seen from Theorem 5, the radical R of the group G is precisely its upper radical $\widetilde{R}(G, z_0)$, constructed by B. I. Plotkin in (10). The question of whether the radical R is also a radical in the sense of A. G. Kurosh (11) remains open. A positive answer to it is obvious only for particular classes of groups, among which are also the most interesting classes with one or another finiteness condition. If it is possible to prove that an arbitrary primary group contains no invariant RI^*-subgroups distinct from the identity, then an affirmative answer will thereby be given in the general case as well.

By **radical groups**, i.e. groups coinciding with their radical, in our case will be meant RI^*-groups. As for semisimple

groups—groups with radical R equal to the identity, then for them one can give the following characterization, following from the definition of R and property b) of the primary normal divisor in § 1.

A group G is semisimple if and only if it contains no abelian normal divisors distinct from the identity.

Such semisimplicity of a group is called F_2-**semisimplicity** (^{8}). Turning to Definition 4, we obtain:

A group G is F_2-semisimple if and only if it is representable as a subdirect product of primary groups.

Hence, and from § 2, the following particular assertions follow:

1. An F_2-semisimple group G with the minimal condition for normal divisors is representable as a subdirect product of subdirectly irreducible primary groups.
2. An F_2-semisimple locally normal group, as well as an F_2-semisimple group with finite classes, is representable as a subdirect product of finite primary groups.
3. An F_2-semisimple group G with the descending chain condition for normal chains is representable as a subdirect product of primary subdirectly irreducible groups whose kernels are completely reducible and consist of a finite number of simple factors.

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