A. I. Sirota

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Abstract

Full Text

A. I. Sirota

KERNELS OF LINEAR REPRESENTATIONS OF NON-COMPACT SIMPLE LIE GROUPS

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According to A. I. Mal' cev ([1]), among locally isomorphic semisimple groups admitting faithful linear representations there is a so-called universal linear group, of which all the others are possible factor groups. It is obtained from the simply connected covering group by factorization with respect to a certain central normal divisor, which we shall call the linearizing one. In the present work the linearizing normal divisors are computed for all noncompact real simple Lie groups (as is known, for compact and semisimple complex groups the universal linear group and the simply connected group coincide, so that the linearizing normal divisor is trivial). Further, using the results of ([2]), the kernels of linear representations of noncompact simple Lie groups are determined.

Let P be a simple compact Lie algebra; $[P]$ its complex form; τ its involutive automorphism. Let G be the real form of $[P]$ singled out by the automorphism τ. This means that $G = P_+ + iP_-$, where $P_+ \subset P$ is the subalgebra belonging to the characteristic root 1 of the automorphism τ, and $P_- \subset P$ is the subspace belonging to the characteristic root -1. By a suitable choice of the Cartan subalgebra H of the algebra P, the automorphism τ can be represented in the form $\tau = \tau_0 \exp(\bar{h})$, where $\bar{h}$ is the matrix of the linear transformation $x \to [x, h]$, $x \in P$, $h \in H$; τ_0 is an involutive automorphism of the algebra P that maps into itself some system $\Pi(P)$ of simple roots of P lying in H, and, moreover, if τ_0 is extended to an automorphism of the whole algebra $[P]$, then $\tau_0(e_\alpha) = e_{\tau_0(\alpha)}$; $\alpha \in \Pi(P)$; e_α is the root vector of $[P]$ corresponding to the root α ([3]). The automorphism τ_0 is either the identity or outer. In the first case G is called a **real form of the first category** of the algebra $[P]$, and in the second case a **real form of the second category**.

If Σ is the full system of roots of P, write the structural formulas of $[P]$ in the form:

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta} \quad (\alpha + \beta \neq 0); \qquad [e_\alpha, e_{-\alpha}] = 2\pi i \alpha;$$

$$[h, e_\alpha] = 2\pi i (h, \alpha) e_\alpha, \qquad h \in H, \quad \alpha, \beta \in \Sigma; \qquad \Sigma \in H.$$

Put $H_+ = H \cap P_+$. H_+ is a Cartan subalgebra for P_+. The compact algebra P_+ admits a decomposition into a direct sum $P_+ = P_1 + V$, where P_1 is compact semisimple and $V \subset H_+$ is a commutative algebra (whose dimension is equal to 0 or 1). Correspondingly $H_+ = H_1 + V$, where H_1 is a Cartan subalgebra of P_1.

For an arbitrary semisimple compact algebra R, we shall denote by $\Gamma_0(R)$ the integral lattice in its Cartan subalgebra with basis

$$\alpha' = \frac{2\alpha}{(\alpha, \alpha)}, \qquad \alpha \in \Pi(R).$$

According to [2], $\Gamma_0(R)$ is the complete inverse image of the identity in the Cartan subalgebra R under the canonical mapping $R \to \mathfrak{R}$, where $\mathfrak{R}$ is the simply connected group with Lie algebra R.

All groups in what follows are assumed to be connected.

Let $\widetilde{G}$ be a simply connected real group with Lie algebra G; $\widehat{G}$ the universal linear group locally isomorphic to $\widetilde{G}$; $\mathfrak{H}_+ \subset \widetilde{G}$, $\widehat{\mathfrak{H}}_+ \subset \widehat{G}$ the subgroups corresponding to the subalgebra H_+. The centers of the groups $\widetilde{G}$ and $\widehat{G}$ lie, respectively, in $\mathfrak{H}_+$ and in $\widehat{\mathfrak{H}}_+$ [4].

Theorem. *The linearizing normal divisor $\mathcal{N}$ of a simply connected simple real group $\widetilde{G}$ is isomorphic to the quotient group $\Gamma_0(P) \cap H_+/\Gamma_0(P_1)$, and the isomorphism is induced by the canonical mapping of the algebra G into the group $\widetilde{G}$.*

Proof. The universal linear group $\widehat{G}$ is contained in a simply connected complex group with Lie algebra $[P]$ [1]. Let $\mathfrak{P}$ be the subgroup of this group corresponding to the subalgebra P; $\mathfrak{P}$ is simply connected and compact. To the subalgebra $H \subset P$ there corresponds in $\mathfrak{P}$ a Cartan subgroup $\mathfrak{H}$ containing $\mathfrak{H}_+$. Let $\tilde{c} : G \to \widetilde{G}$ be the canonical mapping, $\varphi : \widetilde{G} \to \widehat{G}$ the natural homomorphism with kernel $\mathcal{N}$. Then $\varphi(\widetilde{\mathfrak{H}}_+) = \widehat{\mathfrak{H}}_+$, and $\varphi\tilde{c} = \hat{c} : G \to \widehat{G}$ is the canonical mapping. The full inverse image of $\mathcal{N}$ in H_+ under the mapping $\tilde{c}$ coincides with the full inverse image of the identity in H_+ under the mapping $\hat{c}$. The latter coincides with the full inverse image of the identity in H_+ under the canonical mapping $P \to \mathfrak{P}$, i.e. with $\Gamma_0(P) \cap H_+$, since $\widehat{\mathfrak{H}}_+$ belongs to the intersection of $\mathfrak{P}$ and $\widehat{G}$. As shown in [4], the full inverse image of the identity in H_+ under the mapping $\tilde{c}$ is $\Gamma_0(P_1)$ (we note that, generally speaking, the simply connected group corresponding to P_1 does not belong to $\mathfrak{P}$). Since $\mathcal{N} \subset \widetilde{\mathfrak{H}}_+$, we obtain

$$\mathcal{N} \simeq \Gamma_0(P) \cap H_+/\Gamma_0(P_1).$$

Let P_0 be the subalgebra of P belonging to the characteristic root 1 of the automorphism τ_0. P_0 is always semisimple (and compact), and H_+ serves as a Cartan subalgebra for it. Denote by $\Gamma_1(P_0)$ the integral lattice in H_+ whose basis is biorthogonal to the system of simple roots of P_0. It follows from [4] that $\Gamma_1(P_0)$ is the full inverse image of the center of $\widetilde{G}$ in H_+ under the canonical mapping $\tilde{c}$.

It is not difficult to verify that for all simple algebras, with the exception of the algebra of real odd-order matrices of trace 0 (the real form of the second category of type A), $\Gamma_0(P) \cap H_+ = \Gamma_0(P_0)$. Consequently, for all simple simply connected groups, except for the indicated exception, the linearizing normal divisor is isomorphic to $\Gamma_0(P_0)/\Gamma_0(P_1)$.

Corollary 1. *The center $\mathcal{C}(\widehat{G})$ of the universal linear group for a real simple algebra G is isomorphic to the quotient group $\Gamma_1(P_0)/\Gamma_0(P) \cap H_+$, or, except for the exception indicated in the preceding remark, to the quotient group $\Gamma_1(P_0)/\Gamma_0(P_0)$. Thus $\mathcal{C}(\widehat{G})$ can be represented in H_+ as a complete set of representatives of the cosets of $\Gamma_1(P_0)$ modulo $\Gamma_0(P) \cap H_+$.*

In particular, for $\tau_0 = 1$ we have $P_0 = P$, whence

$$\mathcal{C}(\widehat{G}) \simeq \Gamma_1(P)/\Gamma_0(P) \simeq \mathcal{C}(\mathfrak{P}),$$

i.e. the center of the universal linear group for real forms of the first category coincides with the center of the simply connected compact group of the same complex structure, whose representatives in the Cartan subalgebra were computed in ([2]).

Let a linear representation φ of the group $\widehat{G}$ be given by the highest weight Λ of the corresponding linear representation of the algebra $[G]$. The kernel $\mathcal{K}(\varphi)$ of the representation φ is contained in $\mathcal{C}(\widehat{G})$. It is specified by those of the vectors z, representing the elements of $\mathcal{C}(\widehat{G})$ in H_+, for which

$$(z, \Lambda) \equiv 0 \pmod 1$$

([2]).

Below are listed the linearizing normal divisors $\mathcal{N}$ (as subgroups of the center $\mathcal{C}(\widehat{G})$ of the group $\widehat{G}$), the centers of the universal linear groups $\mathcal{C}(\widehat{G})$, and the kernels of linear representations $\mathcal{K}(\varphi)$ for real

simple noncompact groups. On the basis of Corollary 1, it suffices to compute $\mathcal{C}(\hat{G})$ and $\mathcal{K}'(\varphi)$ for groups of the second category ($\tau_0 \neq 1$), since for groups of the first category the problem reduces to the compact groups considered in ([2]). Notation: $Z_m(z)$ is the cyclic group of order m with generator $c(z)$ ($z \in H_+$; c denotes $\hat{c}$ in the computation of $\mathcal{N}$ and $\hat{c}$ in the computation of $\mathcal{C}(\hat{G})$); $Z(z)$ is the infinite cyclic group. With the aid of the vectors $u_i, z, z_i \in H_+$ (see below) the generators of the center $\mathcal{C}(\hat{G})$ are specified; their values are computed in ([4]). The vector Λ is specified by the numerical labels

$$a_i = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)}, \qquad \alpha_i \in \Pi(P)$$

(the simple roots are numbered as in ([2])). The real forms of the exceptional groups are specified by the signature δ of their Cartan metric.

Real forms A_n $(n > 1)$

1. $G = A_n^l$ $(\tau_0 = 1)$—the algebra of matrices of order $n+1$ with trace 0, leaving invariant the Hermitian form

$$-\sum_1^l x_k \overline{y}_k + \sum_{l+1}^{n+1} x_k \overline{y}_k, \qquad l = 1, \dots, \left[\frac{n+1}{2}\right].$$

   $\mathcal{N}$ has the form $Z(u_1 + u_2)$.

2. $G = I_n$ $(\tau_0 \neq 1)$—the algebra of real matrices of order $n+1$ with trace 0.

$\mathcal{N}$ has the form $Z_2(2z)$, if n and $\frac{n+1}{2}$ are odd; $Z_2(z_1)$, if n is odd and $\frac{n+1}{2}$ is even; $Z_2(z_2)$, if n is even. $\mathcal{C}(\hat{G})$ has the form $Z_2(z)$, if n is odd, and is trivial if n is even. $\mathcal{K}'(\varphi)$ is trivial if n is even or if n and $a_1 + a_3 + \dots + a_n$ are odd, and coincides with $\mathcal{C}(\hat{G})$ if n is odd and $a_1 + a_3 + \dots + a_n$ is even.

3. $G = J_n$ (n odd, $\tau_0 \neq 1$)—the algebra of quaternionic matrices of order $\frac{n+1}{2}$, considered up to real positive multipliers.

$\mathcal{N}$ is trivial, $\mathcal{C}(\hat{G})$ coincides with $\mathcal{C}(\hat{\tilde{G}})$ and has the form $Z_2(z)$. $\mathcal{K}'(\varphi)$ is computed in the same way as for I_n.

Real forms B_n

1. $G = B_n^{2l}$ $(\tau_0 = 1)$—the algebra of real matrices of order $2n+1$, leaving invariant the quadratic form

$$-\sum_1^{2l} x_k^2 + \sum_{2l+1}^{2n+1} x_k^2, \qquad l = 1, \dots, n.$$

   $\mathcal{N}$ has the form $Z(z_1)$, if $l = 1$, and $Z_2(z_1)$, if $l > 1$.

Real forms C_n

1. $G = C_n^{2l}$ $(\tau_0 = 1)$—the algebra of matrices of order $2n$, leaving invariant the skew-symmetric bilinear form

$$\sum_1^n (x_{2k-1} y_{2k} - x_{2k} y_{2k-1})$$

   and the Hermitian form

$$-\sum_1^{2l} x_k \overline{y}_k + \sum_{2l+1}^{2n+1} x_k \overline{y}_k.$$

   $\mathcal{N}$ is trivial.

2. $G = JC_n$ $(\tau_0 = 1)$—the algebra of real matrices of order $2n$, leaving invariant the skew-symmetric bilinear form

$$\sum_1^n (x_{2k-1}y_{2k} - x_{2k}y_{2k-1}).$$

$\mathcal{N}$ has the form $Z(2z)$, if n is odd, and $Z(z_1)$, if n is even.

Real forms D_n

1. $G = D_n^{2l}$ $(\tau_0 = 1)$—the algebra of real matrices of order $2n$ leaving invariant the quadratic form

$$-\sum_1^{2l} x_k^2 + \sum_{2l+1}^{2n} x_k^2, \quad l = 1, \dots, \left[\frac{n}{2}\right].$$

$\mathcal{N}$ has the form $Z(2z_1)$, if $l = 1$, n even; $Z(2z_1 + z)$, if $l = 1$, n odd; $Z_2(z_4)$, if $l > 1$, n odd; $Z_2(z_2 - z_1 - z_3)$, if $l > 1$, n and l even; $Z_2(2z_1)$, if $l > 1$, n even, l odd.

2. $G = JD_n(\tau_0 = 1)$—the algebra of matrices of order $2n$ leaving invariant the quadratic form

$$\sum_1^n x_{2k-1}x_{2k}$$

and the Hermitian form

$$\sum_1^n (x_{2k-1}\overline{x}_{2k-1} - x_{2k}\overline{x}_{2k}).$$

$\mathcal{N}$ has the form $Z(4z_1 - 4sz)$, if $n = 2s + 1$; $Z(2z_3 - 4sz)$, if $n = 4s + 2$; $Z(2z_1 - 4sz)$, if $n = 4s$.

3. $G = D_n^{2l+1}$ $(\tau_0 \neq 1)$—the algebra of real matrices of order $2n$ leaving invariant the quadratic form

$$-\sum_1^{2l+1} x_k^2 + \sum_{2l+2}^{2n} x_k^2, \quad l = 0, 1, \dots, [n/2].$$

$\mathcal{N}$ has the form $Z_2(z_4)$. $\mathcal{C}(\tilde{G})$ has the form $Z_2(z)$. $\mathcal{K}'(\varphi)$ coincides with $\mathcal{C}(\tilde{G})$, if $a_{n-1} + a_n$ is odd, and is trivial if $a_{n-1} + a_n$ is even.

Real form G_2

1. $\delta = 2$ $(\tau_0 = 1)$. $\mathcal{N}$ coincides with $\mathcal{C}(\tilde{G})$ and has the form $Z_2(\alpha_2')$.

Real forms F_4

1. $\delta = -52$ ($\tau_0 = 1$). $\mathcal{N}$ is trivial.
2. $\delta = 4$ ($\tau_0' = 1$). $\mathcal{N}$ coincides with $\mathcal{C}(\tilde{G})$ and has the form $Z_2(\alpha_1')$.

Real forms E_6

1. $\delta = -14$ ($\tau_0 = 1$). $\mathcal{N}$ has the form $Z(3z)$.
2. $\delta = 2$ ($\tau_0 = 1$). $\mathcal{N}$ has the form $Z_2(3z)$.
3. $\delta = -26$ ($\tau_0 \neq 1$). $\mathcal{N}$ and $\mathcal{C}(\tilde{G})$ are trivial.
4. $\delta = 6$ ($\tau_0 \neq 1$). $\mathcal{N}$ coincides with $\mathcal{C}(\tilde{G})$ and has the form $Z(z_1)$. $\mathcal{C}(\tilde{G})$ is trivial.

Real forms E_7

1. $\delta = -5$ ($\tau_0 = 1$). $\mathcal{N}$ has the form $Z_2(z_1)$.
2. $\delta = 7$ ($\tau_0 = 1$). $\mathcal{N}$ has the form $Z_2(2z)$.
3. $\delta = -25$ ($\tau_0 = 1$). $\mathcal{N}$ has the form $Z(2z)$.

Real forms E_8

1. $\delta = -24$ ($\tau_0 = 1$). $\mathcal{N}$ coincides with $\mathcal{C}(\tilde{G})$ and has the form $Z_2(\alpha_1)$.
2. $\delta = 8$ ($\tau_0 = 1$). $\mathcal{N}$ coincides with $\mathcal{C}(\tilde{G})$ and has the form $Z_2(\alpha_7)$.

Corollary 2. Among the simply connected noncompact simple real groups, only the groups corresponding to the following algebras have faithful linear representations: J_n (n odd), C_n^{2l}, D_n^1, real forms F_4 with signature -52, E_6 with signature -26.

Corollary 3. All noncompact simple real groups having faithful linear representations have irreducible faithful linear representations, with the exception of the universal linear groups for the algebras D_{2n}^{2l}, JD_{2n}.

Moscow State Pedagogical Institute
named after V. I. Lenin

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References

1. A. I. Mal' tsev, DAN, **40**, No. 3, 108 (1943).

2. E. B. Dynkin, A. L. Onishchik, UMN, **10**, No. 4, 3 (1955).

3. F. R. Gantmakher, Mat. sborn., **5** (47), 101, 217 (1939).

4. A. I. Sirota, DAN, **134**, No. 1 (1960).

Note: Figure translations are in progress. See original paper for figures.

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