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Abstract

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THEORY OF ELASTICITY

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CONTACT PROBLEM FOR AN ELASTIC LAYER UNDER THE ACTION OF AN ECCENTRIC VERTICAL FORCE ON A CIRCULAR RIGID PUNCH

(Presented by Academician Yu. N. Rabotnov on February 20, 1960)

1. For an eccentric force, the solution of the problem is divided into two parts, corresponding to the central force P and to a couple with moment $M = Pe$, where e is the eccentricity (see Fig. 1). As is known, the displacement components can be expressed in the form [1]

$$u = \varphi_1 + z\frac{\partial \varphi_4}{\partial x}, \quad v = \varphi_2 + z\frac{\partial \varphi_4}{\partial y}, \quad w = \varphi_3 + z\frac{\partial \varphi_4}{\partial z}, \tag{1}$$

where all four functions φ_i $(i = 1, 2, 3, 4)$ are harmonic and are related by

$$\frac{\partial \varphi_4}{\partial z} = -\frac{1}{3-4\nu}\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} + \frac{\partial \varphi_3}{\partial z}\right).$$

For a centrally applied vertical force, corresponding to the axisymmetric problem, the following may be taken as the initial harmonic functions [2]:

$$\varphi_1 = \frac{x}{r}\int_0^\infty [A\,\mathrm{sh}(z\alpha) + B\,\mathrm{ch}(z\alpha)]J_1(r\alpha)\,d\alpha,$$

$$\varphi_2 = \frac{y}{r}\int_0^\infty [A\,\mathrm{sh}(z\alpha) + B\,\mathrm{ch}(z\alpha)]J_1(r\alpha)\,d\alpha, \tag{2}$$

$$\varphi_3 = \int_0^\infty [C\,\mathrm{sh}(z\alpha) + D\,\mathrm{ch}(z\alpha)]J_0(r\alpha)\,d\alpha.$$

Fig. 1. Displacement of a circular punch under eccentric loading

Figure 1: Fig. 1. Displacement of a circular punch under eccentric loading

If one uses the solution of the contact problem for an elastic half-space [3], then, as the initial harmonic functions in the case of the action of a moment, one must take

$$\varphi_1 = \int_0^\infty [A\,\mathrm{sh}(z\alpha) + B\,\mathrm{ch}(z\alpha)] \left[\frac{x^2}{r^2} J_2(r\alpha) - \frac{J_1(r\alpha)}{r\alpha}\right] d\alpha,$$

$$\varphi_2 = \int_0^\infty [A\,\mathrm{sh}(z\alpha) + B\,\mathrm{ch}(z\alpha)] \frac{xy}{r^2} J_2(r\alpha)\, d\alpha, \tag{3}$$

$$\varphi_3 = \int_0^\infty [C\,\mathrm{sh}(z\alpha) + D\,\mathrm{ch}(z\alpha)] \frac{x}{r} J_1(r\alpha)\, d\alpha.$$

Here it is assumed that the direction of the moment coincides with the x-axis. In equalities (2) and (3), $J_0(r\alpha)$, $J_1(r\alpha)$, $J_2(r\alpha)$ are Bessel functions of the first kind, and the unknown coefficients A, B, C, D depend on the integration parameter α and are determined from the prescribed boundary conditions.

2. Let the elastic layer have thickness H, modulus of elasticity E, and Poisson' s ratio ν. To simplify the solution of the problem it is assumed that friction between the punch and the layer, and also between the layer and the fixed base, is absent, although the original equalities (2) and (3) make it possible to solve this problem under other boundary conditions.

It is easy to see from (1) that, for the adopted system of coordinate axes, we have $(w)_{z=0} = 0$ when $D = 0$. Using the components of the tangential stresses expressed in terms of the harmonic functions φ_i $(i = 1, 2, 3, 4)$, one can show that, in the absence of friction on the lower boundary of the layer, we have $A = 0$. In addition, the absence of friction under the punch leads to the relation

Fig. 1. Displacement of a circular punch under eccentric loading

$$C = \frac{2(1-\nu)\,\mathrm{sh}(\alpha H) + \mathrm{ch}(\alpha H)}{(1-2\nu)\,\mathrm{sh}(\alpha H) - (\alpha H)\,\mathrm{ch}(\alpha H)}\, B.$$

For convenience of exposition, let us write simply α instead of αH, and also

$$B(\alpha) = \frac{1+\nu}{E} \frac{(1-2\nu)\,\mathrm{sh}\,\alpha - \alpha\,\mathrm{ch}\,\alpha}{\mathrm{sh}^2\,\alpha} M(\alpha).$$

The coefficient $M(\alpha)$ is determined from the components, prescribed on the upper boundary of the elastic layer, of the normal stress σ_z and the vertical displacement w, which are expressed as follows:

$$\sigma_z = \frac{1}{H}\int_0^\infty \frac{\alpha M(\alpha)}{1-g(\alpha)}\left(\frac{x}{r}\right)^m J_m\left(\frac{r}{H}\alpha\right)d\alpha,$$

$$w = \frac{2(1-\nu^2)}{E}\int_0^\infty M(\alpha)\left(\frac{x}{r}\right)^m J_m\left(\frac{r}{H}\alpha\right)d\alpha. \tag{4}$$

Here, for $m = 0$, we have σ_z and w for a centrally applied force, and for $m = 1$, for a moment. In addition, in equality (4) it is assumed that

$$g(\alpha) = 1 - \frac{\operatorname{sh}^2\alpha}{\alpha + \operatorname{sh}\alpha\operatorname{ch}\alpha}. \tag{5}$$

The transition to the half-space is carried out by replacing α by $(H\alpha)$. Then, for $H = \infty$, we have $g(\alpha) = 0$.

3. For the paired integral equations (4), the solution for $m = 0$ is available in the work of N. N. Lebedev and Ya. S. Uflyand ([4]). The solution of these same equations for $m = 1$ may be sought in the form

$$M(\alpha) = [1-g(\alpha)]\frac{1}{H}\int_0^R \varphi(t)\sin\left(\frac{\alpha}{H}t\right)dt, \tag{6}$$

where $\varphi(t)$, together with its first derivative, is assumed continuous on the interval $(0, R)$, and it is assumed that $\varphi(0) = 0$.

The solution of (4) for $m = 1$ is considered with boundary conditions corresponding to the action of a moment:

$$\sigma_z = 0 \quad (r > R), \qquad w = kx \quad (r < R).$$

Here k is the tangent of the angle of inclination of the punch. The generality of the subsequent arguments remains valid also when $x = r\cos\theta$, where θ is the angle between the axes x and r (see Fig. 1).

Substituting (6) into equality (4) for $m = 1$ and taking into account (5), that

$$\int_0^\infty \alpha\sin\left(\frac{t}{H}\alpha\right)J_1\left(\frac{r}{H}\alpha\right) = 0 \qquad (t \le R < r),$$

we obtain outside the punch $\sigma_z = 0$.

Integrating the left-hand side of equality (6) by parts and substituting the resulting expression into equality (4), for $m = 1$ we have

$$\sigma_z = \frac{1}{H}\frac{x}{r}\left[\int_0^R \varphi'(t)\,dt\int_0^\infty \cos\left(\frac{t}{H}\alpha\right)J_1\left(\frac{r}{H}\alpha\right)d\alpha - \varphi(R)\int_0^\infty \cos\left(\frac{R}{H}\alpha\right)J_1\left(\frac{t}{H}\alpha\right)d\alpha\right]. \tag{7}$$

On the basis of the known formulas (5)

$$\int_0^\infty \cos\left(\frac{R}{H}\alpha\right)J_1\left(\frac{r}{H}\alpha\right)d\alpha = \frac{H}{r}\left(1 - \frac{R}{\sqrt{R^2-r^2}}\right) \qquad (r < R);$$

$$\int_0^\infty \cos\left(\frac{t}{H}\alpha\right)J_1\left(\frac{r}{H}\alpha\right)d\alpha = \frac{H}{r} \qquad (r > t);$$

$$\int_0^\infty \cos\left(\frac{t}{H}\alpha\right)J_1\left(\frac{r}{H}\alpha\right)d\alpha = \frac{H}{r}\left(1 - \frac{t}{\sqrt{t^2-r^2}}\right) \qquad (r < t)$$

equality (7) takes the form

$$\sigma_z = \frac{x}{r^2}\left[\frac{R\varphi(R)}{\sqrt{R^2-r^2}} - \int_r^R \frac{t\varphi'(t)}{\sqrt{t^2-r^2}}\,dt\right]. \tag{8}$$

The function $\varphi(t)$ is determined with account taken of the displacement of the punch prescribed on the boundary of the elastic layer.

Substituting into (4) $w = kx$, we obtain for $m = 1$

$$fr = \int_0^\infty M(\alpha)J_1\left(\frac{r}{H}\alpha\right)d\alpha, \tag{9}$$

where $f = Ek/2(1-\nu^2)$.

Substitution of $M(\alpha)$ from (6) into (9) and the use of the known formulas (5)

$$\int_0^\infty \sin\left(\frac{t}{H}\alpha\right)J_1\left(\frac{r}{H}\alpha\right)d\alpha = \begin{cases} \dfrac{Ht}{r\sqrt{t^2-r^2}} & (0 \le t < r); \\ 0 & (t > r); \end{cases}$$

$$J_1\left(\frac{r}{H}\alpha\right) = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,\sin\left(\frac{r\alpha}{H}\sin\theta\right)d\theta$$

leads to the relation

$$fr^2 = \int_0^r \frac{t\varphi(t)}{\sqrt{t^2 - r^2}}\, dt - \frac{2}{\pi H} \int_0^{\pi/2} r \sin\theta \, d\theta \int_0^R \varphi(t)\, dt \int_0^\infty g(\alpha) \sin\left(\frac{t}{H}\alpha\right) \sin\left(\frac{r\alpha}{H}\sin\theta\right) d\alpha.$$

If in the first integral we make a change of variable and denote

$$K(t \pm x) = \frac{R}{H} \int_0^R g(\alpha) \cos\left[(t \pm x)\frac{\alpha}{H}\right] d\alpha. \tag{10}$$

we arrive at the integral equation of Schlömilch

$$fr^2 = \int_0^{\pi/2} r \sin\theta \, F(r \sin\theta)\, d\theta, \tag{11}$$

where

$$\varphi(x) + \frac{1}{\pi R} \int_0^R [K(t + x) - K(t - x)]\varphi(t)\, dt = F(x). \tag{12}$$

Using the known solution of equation (11) ([6]), we obtain

$$F(x) = \frac{4}{\pi} fx = \frac{2}{\pi} \frac{Ekx}{1 - \nu^2}. \tag{13}$$

4. For numerical calculations it is convenient to pass to dimensionless quantities:

$$\frac{x}{R} = \xi, \qquad \frac{t}{R} = \tau, \qquad \varphi(\xi) = \frac{2REk}{\pi(1 - \nu^2)}\omega(\xi). \tag{14}$$

Then the Fredholm integral equation (12), taking (13) into account, assumes the form

$$\omega(\xi) = \xi - \frac{1}{\pi} \int_0^1 [K(\tau + \xi) - K(\tau - \xi)]\omega(\tau)\, d\tau. \tag{15}$$

Equality of the moments of the external force and of the reactive pressures under the punch leads to the relation

$$Pe = 2\pi \int_0^R t\varphi(t)\, dt.$$

Substituting into this equality the dimensionless quantities from (14), we obtain, for the angle of inclination of the circular punch under the action of a moment, the formula

$$k = \frac{(1-\nu^2)Pe}{4R^3E\int_0^1 \xi\omega(\xi)\,d\xi}.$$

The reactive pressure under the circular punch due to the action of the moment $M = Pe$ is determined by formula (8), which, taking account of the dimensionless quantities (14), assumes the form:

$$\sigma_z = \frac{Pe}{2\pi R^2\int_0^1 \xi\omega(\xi)\,d\xi}\left[\frac{\omega(1)}{\sqrt{1-(r/R)^2}} + \int_1^{r/R}\frac{\tau\omega'(\tau)\,d\tau}{\sqrt{\tau^2-(r/R)^2}}\right]\frac{x}{r^2}.$$

As the computation carried out in calculating circular foundations on a compressible base has shown, it is sufficient to represent $\omega(\tau) = a_1\tau + a_3\tau^3$. The coefficients a_1, a_3 are determined from the Fredholm integral equation (15); moreover, for a half-space, when $H = \infty$, we have $\omega(\tau) = \tau$. In the computation we used for (5) the approximation (7) in the form $\sum B_i e^{-A\alpha}$.

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Note: Figure translations are in progress. See original paper for figures.