PHYSICS

V. V. TOLMACHEV

1960

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Abstract

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PHYSICS

V. V. TOLMACHEV

TEMPERATURE ELEMENTARY EXCITATIONS IN A NONIDEAL BOSE-EINSTEIN SYSTEM

(Presented by Academician N. N. Bogolyubov, 17 VI 1960)

We shall use a modified formulation of the problem, according to which the operators a_0^+ and a_0 for the creation and annihilation of particles with zero momentum are replaced by the c-number $\sqrt{N^0}$ (see ([1,2])). In this formulation the Fourier component of the density operator ρ_q, with momentum q, is decomposed into two components: one describing the change in the density of overcondensate particles arising from processes of creation and annihilation of particles by the condensate, and the other describing the true change in the density of overcondensate particles,

$$\rho_q^0 = \sqrt{N_0}\,(a_q + a_{-q}^+), \qquad \rho_q' = \sum_p a_p^+ a_{p+q}, \tag{1}$$

where the summation is carried out over all momenta except $p = 0$ and $p = -q$.

Accordingly, we shall consider the following temperature Green' s functions, the poles of which will give us the temperature elementary excitations,

$$D(q; t-t') = \frac{i}{\hbar}\left\langle T(\rho_q^0(t)\rho_{-q}^0(t'))\right\rangle,$$

$$Q(q; t-t') = \frac{i}{\hbar}\left\langle T(\rho_q'(t)\rho_{-q}'(t'))\right\rangle, \tag{2}$$

where the transition to time-dependent operators, as well as the subsequent averaging of chronological products of these operators, is performed with the aid of the full operator $\Omega = H - \mu N$, where H is the Hamiltonian operator, N is the operator of the total number of particles, and μ is the chemical potential. Thus:

$$\rho_q^0(t) = e^{\frac{i}{\hbar}\Omega t}\rho_q^0 e^{-\frac{i}{\hbar}\Omega t}, \qquad \rho_q'(t) = e^{\frac{i}{\hbar}\Omega t}\rho_q' e^{-\frac{i}{\hbar}\Omega t},$$

$$\langle\cdots\rangle = \mathrm{Sp}\, e^{-\beta\Omega}\cdots \;/\; \mathrm{Sp}\, e^{-\beta\Omega},$$

where $\beta = 1/\theta$, $\theta = kT$, k is Boltzmann' s constant, and T is the absolute temperature.

In notes ([1,2]) a method was developed for finding the principal part of the operator Ω. For it we have

$$\Omega_0 = \sum_p \varepsilon(p) b_p^+ b_p. \tag{3}$$

The new Bose operators entering into (3) are related to the original Bose operators by Bogolyubov' s canonical transformation,

$$a_p = u_p b_p + v_p b_{-p}^+. \tag{4}$$

In ([1,2]) a system of nonlinear integral equations is given for determining u_p, v_p, and $\varepsilon(p)$ (see (7) from ([1]) and (1), (2), (3) from ([2])).

In the zeroth approximation we can replace in (2) the operator Ω by the operator Ω_0. Then in the energy representation we obtain

$$D_0(q;E) = N_0(u_q + v_q)^2 \left\{ \frac{1}{-E+\varepsilon(q)} + \frac{1}{E+\varepsilon(q)} \right\}; \tag{5}$$

$$\begin{aligned} Q_0(q;E) = &-\frac{1}{2}\sum_p (u_{p-q}u_p + v_{p-q}v_p)^2 (n_{p-q} - n_p)\times \\ &\times \left\{ \frac{1}{-E+\varepsilon(p-q)-\varepsilon(p)} + \frac{1}{E+\varepsilon(p-q)-\varepsilon(p)} \right\} + \\ &+ \sum_p (u_{p-q}v_p + u_{pv_{p-q}})^2 (1+2n_p)\times \\ &+ \left\{ \frac{1}{-E+\varepsilon(p-q)+\varepsilon(p)} + \frac{1}{E+\varepsilon(p-q)+\varepsilon(p)} \right\}. \end{aligned} \tag{6}$$

The propagator (5) describes the propagation of an individual excitation; the propagator (6) describes the propagation of an unbound pair of such excitations. According to (5) and (6), the energy of an individual excitation is $\varepsilon(p)$.

Wishing to study the collective excitation associated with pairs of individual excitations, we, following the usual arguments used in the theory of collective excitations in a nonideal Fermi-Dirac system (see, for example, ([3,4])), must write the following secular equation:

$$1+\frac{\nu(q)}{V}Q_0(q;E)=0. \tag{7}$$

Taking into account the interaction of collective and individual excitations through the triple vertices of the Hamiltonian, we must write a more exact secular equation

$$1-\frac{\nu^2(q)}{V^2}D_0(q;E)\frac{Q_0(q;E)}{1+\frac{\nu(q)}{V}Q_0(q;E)}=0. \tag{8}$$

Let us now indicate the characteristic temperature dependence of the spectrum of individual excitations, which follows from the study of the system of equations (1), (2), and (3) from ([2]). Finding for $A(p)$ and $B(p)$ the principal asymptotic form at small ν and substituting it into (3) from ([2]), we obtain

$$\varepsilon^2(p)=E^2(p)+2n_0\nu(p)E(p), \tag{9}$$

which is the temperature generalization of the known spectrum of elementary excitations for zero temperature ([5]). For large $|p|$, according to (8), $\varepsilon(p)$ goes over into the individual excitation for an ideal Bose-Einstein system

$$\varepsilon(p)=E(p). \tag{10}$$

For $|p|\ll\nu^{1/2}$, according to (9), $\varepsilon(p)$ has a phonon character

$$\varepsilon(p)=c|p|,\qquad c^2=\frac{\hbar^2}{m}n_0\nu(0). \tag{11}$$

For temperatures above the critical one, n_0 becomes zero. Thus the velocity c vanishes at temperatures above the critical one.

A more detailed study of the system of equations (1), (2), and (3) from ([2]) shows that for very small ν, when $|p|\ll\nu^{3/4}$, from the higher terms in the asymptotics of $A(p)$ and $B(p)$ at small ν there arises a characteristic behavior of $\varepsilon(p)$, different from phonon behavior; namely, it turns out that $\varepsilon(p)$ has a gap

$$\varepsilon^2(0)=\frac{2}{\pi\hbar^3}m^{3/2}n_0^{3/2}\nu^{5/2}(0)\theta. \tag{12}$$

Let us note that, according to (12), the gap vanishes at zero temperature. It also disappears at temperatures above the critical one.

However, formula (12) poorly represents the behavior of the gap at temperatures in the immediate vicinity of zero and of the critical temperature, since the corresponding asymptotic expansions $A(p)$ and $B(p)$ for small v converge nonuniformly in these regions. Studying the behavior of the system (1), (2), and (3) from ([2]) at $\theta = 0$ and for small v, one can obtain that for $\theta = 0$

$$\varepsilon^2(0) = 2n_0^2 v(0) \frac{1}{V} \sum_p \frac{v^2(p)}{E(p)} \tag{13}$$

(which coincides with the result obtained in ([6])) and that for small n_0

$$\varepsilon^2(0) = 4n_0^2 v^2(0). \tag{14}$$

The question may arise whether the results obtained by us contradict completely general arguments (see ([7,8])) to the effect that, absolutely rigorously, one must have

$$D(q; E)\big|_{E=0} > \text{const} \cdot \frac{1}{q^2},$$

from which one would like to conclude that the spectrum of elementary excitations cannot have a gap. The function $D_0(q; E)$ from (5) does not satisfy the indicated inequality. However, the first-approximation function, which differs from $D_0(q; E)$ by the presence of a factor in whose denominator stands the expression from the left-hand side of (8), precisely in view of this factor, will satisfy the inequality. The function $D(q; E)$ has two poles: one corresponding to an individual excitation and one corresponding to a collective excitation. These poles are determined from the secular equation (8).

To study the spectrum of collective excitations, let us pass in (7) from the sum to an integral and make the rough assumption that $\varepsilon(p)$ may be replaced by $E(p)$, i.e., we shall neglect in (7) the effect of the canonical transformation (4). Then instead of (7) we obtain*

$$1 + \left(1 - \left(\frac{\theta}{\theta_0}\right)^{3/2}\right) nv(q) \left\{\frac{1}{E(q) - E} + \frac{1}{E(q) + E}\right\} +$$

$$+ \frac{v(q)}{(2\pi)^3} \int dp \, \frac{n_p - n_{p+q}}{E - E(p) + E(p+q)} = 0, \tag{15}$$

where n_p are the occupation numbers for the ideal Bose-Einstein system. Setting in (15) $E = s|q|$ and passing to the limit $q \to 0$, we obtain the following transcendental equation for determining s:

$$s^2 = \left(1 - \left(\frac{\theta}{\theta_0}\right)^{3/2}\right) \frac{\hbar^2}{m} nv(0) \times$$

$$\times \left[1 - \frac{v(0)}{(2\pi)^2}\frac{1}{\theta}\int_0^{+\infty} p^2 dp\, \frac{e^{\hbar^2 p^2/2m\theta}}{(e^{\hbar^2 p^2/2m\theta}-1)}\left\{2 + \frac{sm}{\hbar^2 p}\ln\left|\frac{s-\hbar^2 p/m}{s+\hbar^2 p/m}\right|\right\}\right]^{-1},$$

whence we obtain, if we substitute for the integral in (16) its asymptotic value at large s:

$$s^2 = \frac{\hbar^2}{m} n v(0), \tag{17}$$

an expression independent of temperature.

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* In fact, the dimensionless small parameter characterizing the approximation leading to (15) is $q_0/\sqrt{m\theta_0}$, where q_0 is the size of the potential $v(q)$ in momentum space, and θ_0 is the critical temperature for the ideal Bose-Einstein system.

The scheme we have proposed requires improvement in the case of singular interaction potentials of the hard-sphere interaction type. This improvement amounts to replacing the $v(0)$ appearing in the formulas by the corresponding matrix elements of the scattering operator $f(0) = \frac{\hbar^2}{2m} 4\pi a$, where a is the so-called scattering length (in the case of hard-sphere interaction, the radius of the particle).

We draw attention to the fact that in the immediate vicinity of the critical temperature our approximations lose their validity. Therefore, in particular, within the scheme developed here we cannot obtain any information about the nature of the singularity of thermodynamic quantities at the critical temperature.

Note added in proof. Although the arguments given above seem to indicate with sufficient persuasiveness the presence of two branches in the spectrum of thermal elementary excitations, it may nevertheless turn out that a more detailed study of the secular equation (8) will show that in fact we are dealing with a single branch of the spectrum, and that our separation of the excitations into individual and collective ones will have only the following meaning: whether we pay greater attention to the behavior of the elementary excitation at large momenta (individual excitation) or at small momenta (collective excitation). In addition, independently of this circumstance, equation (8) may lead to the closing of the gap obtained in the approximation adopted here.

Physical-Chemical Institute
named after L. Ya. Karpov

Received
11 VI 1960

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Note: Figure translations are in progress. See original paper for figures.