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Abstract

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A. B. VASIL' EVA

ASYMPTOTICS OF SOLUTIONS OF CERTAIN BOUNDARY-VALUE PROBLEMS FOR EQUATIONS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

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In previous papers ([1,2]) asymptotic formulas were given for the solution of the problem with initial conditions for a system of the form

$$\mu \frac{dz}{dt} = F(z, y, t),$$

$$\frac{dy}{dt} = f(z, y, t); \tag{1}$$

$$z\big|_{t=t^0} = z^0, \qquad y\big|_{t=t^0} = y^0 \tag{2}$$

($z = \{z_1, \ldots, z_M\}$ is an M-component vector, $y = \{y_1, \ldots, y_m\}$ is an m-component vector), if the parameter $\mu > 0$ tends to zero. There it was also indicated that these formulas could be applied to the construction of asymptotics of solutions of certain boundary-value problems, which was carried out for the case of a second-order equation. In the present note it will be shown how the methods developed in ([2]) apply to more complicated cases.

Let us prescribe for (1) the boundary conditions (R is an $(M + m)$-dimensional vector)

$$R(y(0),\, y(1),\, z(0),\, z(1)) = 0. \tag{3}$$

In ([2]) it was shown that the solution of the boundary-value problem for the equation $\mu y'' = F(y', y, t)$ may, as $\mu \to 0$, possess qualitatively different limiting properties; namely, two characteristic types of limiting behavior of the solution were noted. In the present note, conditions will be formulated under which, also in the general case of the system (1) and the boundary conditions (3), there exists a solution of type I, when the limiting functions corresponding to y are continuous on $[0, 1]$ and y tends to them uniformly, while the limiting functions corresponding to z have a discontinuity at one of the endpoints $t = 0$ or $t = 1$; or of type II, when each of the limiting functions corresponding to y has a corner point at $t = t^0$ $(0 < t^0 < 1)$, and the limiting function corresponding to z (the conditions will be given for $M = 1$, since otherwise solutions of this type, generally speaking, do not exist) has a discontinuity of the first kind at $t = t^0$.

I. Let the system of equations $F(z, y, t) = 0$ have a stable root $z = \varphi(y, t)$, defined in some closed bounded domain $D(y, t)$. We construct two auxiliary systems corresponding to (1):

$$\frac{d\overset{(1)}{z}}{d\tau} = F(\overset{(1)}{z}, \overset{(1)}{y}, \tau\mu)$$

$$\frac{d\overset{(1)}{y}}{d\tau} = \mu f(\overset{(1)}{z}, \overset{(1)}{y}, \tau\mu); \qquad (\tau = t/\mu) \tag{4A}$$

$$\begin{aligned} \mu\frac{d\overset{(2)}{z}}{dt} &= F(\overset{(2)}{z}, \overset{(2)}{y}, t), \\ \frac{d\overset{(2)}{y}}{dt} &= f(\overset{(2)}{z}, \overset{(2)}{y}, t), \end{aligned} \tag{4B}$$

and their formal solutions (for brevity, we shall denote z and y collectively by x)

$$\begin{aligned} \overset{(1)}{x} &= \overset{(1)}{x}_0(\tau) + \mu \overset{(1)}{x}_1(\tau) + \dots, \\ \overset{(2)}{x} &= \overset{(2)}{x}_0(t) + \mu \overset{(2)}{x}_1(t) + \dots, \\ \overset{(2)}{\bar{x}} &= \overset{(2)}{\bar{x}}_{00} + \mu \overset{(2)}{\bar{x}}_{01} + t \overset{(2)}{\bar{x}}_{10} + \dots \end{aligned} \tag{5}$$

The coefficients of these expansions will satisfy systems of equations obtained as a result of substituting (5) into (4), and the initial conditions, which are determined as follows. Substitute (5) into (3), after which we shall also represent R in the form of a formal expansion

$$R(\overset{(1)}{y}_0(0) + \mu \overset{(1)}{y}_1(0) + \dots,\ \overset{(2)}{y}_0(1) + \mu \overset{(2)}{y}_1(1) + \dots,\ \overset{(1)}{z}_0(0) + \mu \overset{(1)}{z}_1(0) + \dots,$$

$$\overset{(2)}{z}_0(1) + \mu \overset{(2)}{z}_1(1) + ...) = R_0 + \mu R_1 + ... = 0.$$

In the zeroth approximation we obtain from this

$$R_0 = R(\overset{(1)}{y}_0(0), \overset{(2)}{y}_0(1), \overset{(1)}{z}_0(0), \overset{(2)}{z}_0(1)) = 0. \tag{6}$$

In these equations one may regard $\overset{(1)}{y}_0(0)$ and $\overset{(1)}{z}_0(0)$ as unknowns (since $\overset{(2)}{y}_0(1)$, $\overset{(2)}{z}_0(1)$ are expressed through $\overset{(2)}{y}_0(0)$ from the system of differential equations defining them, and $\overset{(2)}{y}_0(0) = \overset{(1)}{y}_0(0)$ according to (2)). Suppose that equations (6) are solvable with respect to $\overset{(1)}{y}_0(0)$, $\overset{(1)}{z}_0(0)$, and that the corresponding functional determinant $\Delta \neq 0$. Having determined $\overset{(1)}{y}_0(0)$, $\overset{(1)}{z}_0(0)$, one can then, successively putting $R_k = 0$ $(k = 1, 2, ...)$ and using the general relation

$$\overset{(2)}{y}_k(0) = \overset{(1)}{y}_k(0) + \frac{(-1)^k}{k!} \int_0^\infty \tau^k \frac{d^k}{d\tau^k} \overset{(1)}{f}_{k-1} \, d\tau,$$

determine from these equations $\overset{(1)}{z}_k(0)$, $\overset{(1)}{y}_k(0)$ (and consequently also $\overset{(2)}{y}_k(0)$), and thereby determine the coefficients of the expansions (5).

Let us form, from the coefficients of the expansions (5), the combinations

$$X_n = (\overset{(1)}{x})_n + (\overset{(2)}{x})_n - (\overset{(2)}{\bar{x}})_n,$$

where each of the three summands is the partial sum of the expansions (5), in the order in which they are written, containing terms up to order n inclusive, with respect to μ or with respect to t and μ.

Theorem. Suppose the system $F(z, y, t) = 0$ has a root $z = \varphi(y, t)$, stable in some domain D; suppose the system (6) is solvable with respect to $\overset{(1)}{y}_0(0)$, $\overset{(1)}{z}_0(0)$, and that the corresponding functional determinant $\Delta \neq 0$; suppose, further, that the curve $y = \overset{(2)}{y}_0(t)$ belongs to D for $0 \leq t \leq 1$, and that the point $\overset{(1)}{z}_0(0), \overset{(1)}{y}_0(0), 0$ belongs to the domain of influence of the root $z = \varphi(y, t)$. Then one can indicate a sufficiently small, but μ-independent, neighborhood of the point $\overset{(1)}{y}_0(0)$, $\overset{(1)}{z}_0(0)$, in which, for every $\mu \leq \mu^0$, there exists a unique value y^0, z^0 such that the solution $X(t, \mu)$ of system (1), satisfying the initial ···

initial conditions $X\big|_{t=0} = x^0$, also satisfies the boundary condition (3). If the right-hand sides of system (1) have continuous partial derivatives up to order $(n+1)$ inclusive, and F_z and f_z have continuous partial derivatives up to orders $2n$ and, respectively, $(2n-1)$ inclusive,* then the inequalities

$$|X(t,\mu) - X_n| < C\mu^{n+1},$$

hold, where C is a constant independent of t, μ for $\mu \leqslant \mu^0,\ 0 \leqslant t \leqslant 1$.

II. Let the equation $F(z,y,t) = 0$ have a stable root $z = \varphi^+(y,t)$ and an unstable root $z = \varphi^-(y,t)$, defined in some closed bounded domain $D(y,t)$. Construct the auxiliary systems corresponding to (1):

$$\begin{aligned} \frac{d\overset{(1)}{z}}{d\tau} &= \overset{(1)}{F}(\overset{(1)}{z}, y, t^0 + \tau\mu), \\ \frac{d\overset{(1)}{y}}{d\tau} &= \mu \overset{(1)}{f}(\overset{(1)}{z}, y, t^0 + \tau\mu); \end{aligned} \tag{7A}$$

$$\begin{aligned} \mu\frac{d\overset{(2)}{z}}{dt} &= \overset{(2)}{F}(\overset{(2)}{z}, y, t), \\ \frac{d\overset{(2)}{y}}{dt} &= \overset{(2)}{f}(\overset{(2)}{z}, y, t) \end{aligned} \tag{7B}$$

and their formal solutions

$$\begin{aligned} \overset{(1)}{x} &= \overset{(1)}{x}_0(\tau) + \mu \overset{(1)}{x}_1(\tau) + \dots, \\ \overset{(2)}{x}{}^{\pm} &= \overset{(2)}{x}{}^{\pm}_0(t) + \mu \overset{(2)}{x}{}^{\pm}_1(t) + \dots, \\ \overset{(2)}{x}{}^{\pm} &= \overset{(2)}{x}{}^{\pm}_{00} + \mu \overset{(2)}{x}{}^{\pm}_{01} + (t - t^0)\overset{(2)}{x}{}^{\pm}_{10} + \dots . \end{aligned} \tag{8}$$

The sign + or − indicates that φ^+ or φ^- participates in the construction of the systems of differential equations determining the coefficients of the expansions (8).

We obtain the initial conditions for determining the coefficients of the expansions (8) as follows. Substitute (8) into (3), after which R is also represented in the form of a formal expansion

$$R(\overset{(2)}{y}{}^-_0(0) + \mu \overset{(2)}{y}{}^-_1(0) + \dots, \overset{(2)}{y}{}^+_0(1) + \mu \overset{(2)}{y}{}^+_1(1) + \dots, \overset{(2)}{z}{}^-_0(0) + \mu \overset{(2)}{z}{}^-_1(0) + \dots,$$
$$\overset{(2)}{z}{}^+_0(1) + \overset{(2)}{z}{}^+_1(1) + \dots) = R_0 + \mu R_1 + \dots = 0.$$

In the zero approximation we have from this

$$R_0 = R(\overset{(2)}{y}{}^-_0(0), \overset{(2)}{y}{}^+_0(1), \overset{(2)}{z}{}^-_0(0), \overset{(2)}{z}{}^+_0(1)) = 0. \tag{9}$$

In these equations we shall regard as unknowns $\overset{(1)}{y}_0(0) = \overset{(2)}{y}_0(t^0)$ and t^0. Suppose that they are solvable with respect to these unknowns and that the corresponding determinant $\Delta \neq 0$. Then, putting $R_k = 0$ $(k = 1, 2, ...)$ and using the general relation

$$\overset{(2)}{y}{}^{\pm}_{k}(t^0) = \overset{(1)}{y}_k(0) - \frac{(-1)^k}{k!} \int_0^{\pm\infty} \tau^k \frac{d^k}{d\tau^k} \overset{(1)}{f}_{k-1}\, d\tau,$$

* It is sufficient that the indicated requirements be satisfied in some arbitrarily small neighborhood of the cur

besides, the requirements concerning F_z and f_z can be weakened. An analogous remark may be made concerning the requirements stated in the theorem of item II.

we shall each time obtain systems of equations for the unknowns $y_k^{(1)}(0)$, $z_{k-1}^{(1)}(0)$, and, beginning with $k = 2$, these systems will be linear. Under the assumption that for $k = 1$ the system is solvable, the solvability of the subsequent systems follows automatically by virtue of their linearity (since the nonvanishing of the functional determinants of all these systems follows from the condition $\Delta \neq 0$). Having determined $y_k^{(1)}(0)$, $z_{k-1}^{(1)}(0)$, we thereby determine $y_k^{(2)\pm}(t^0)$, and consequently all coefficients of the expansions (8).

Let us form from these coefficients the combinations

$$X_n = \begin{cases} X_n^-, & 0 \leqslant t \leqslant t^0, \\ X_n^+, & t^0 \leqslant t \leqslant 1, \end{cases}$$

where $X_n^{\pm}$ are composed from the coefficients of the expansions (8) according to the same rule as X_n in case I from the coefficients of the expansions (5).

Theorem. Let the equation $F(z, y, t) = 0$ have a stable root $z = \varphi^+(y, t)$ and an unstable root $z = \varphi^-(y, t)$, defined in the domain D; let the system $R_0 = 0$ be solvable with respect to $y_0^{(1)}(0)$, t^0, where t^0 belongs to $(0, 1)$, and the system $R_1 = 0$ be solvable with respect to $y_1^{(1)}(0)$, $z_0^{(1)}(0)$, with

$$\varphi^-(y_0^{(1)}(0), t^0) \geqslant z_0^{(1)}(0) \geqslant \varphi^+(y_0^{(1)}(0), t^0);$$

let $\Delta \neq 0$; finally, let the curve $y = Y_0$ belong to D for $0 \leqslant t \leqslant 1$, and the point $z_0^{(1)}(0)$, $y_0^{(1)}(0)$, t^0 belong to the domain of influence of both roots φ^+, φ^-. Then one can specify a μ-independent neighborhood of the point $z_0^{(1)}(0)$, $y_0^{(1)}(0)$, in which, for every sufficiently small $\mu \leqslant \mu^0$, there exists a unique value x^0 such that the solution $X(t, \mu)$ of the system (1), satisfying the initial condition $X|_{t=t^0} = x^0$, also satisfies the boundary condition (3). If the right-hand sides of the system (1) possess continuous partial derivatives up to order $(n+1)$ inclusive, and F_z and f_z possess continuous partial derivatives respectively up to orders $2n$ and $(2n-1)$ inclusive, then the inequalities

$$|X(t,\mu) - X_n| < C\mu^{n+1},$$

hold, where C is some constant independent of μ and t for $\mu \leqslant \mu^0$, $0 \leqslant t \leqslant 1$.

Moscow State University
named after M. V. Lomonosov

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1. A. B. Vasil' eva, *Matem. sborn.*, **50** (92), No. 1 (1960). 2. A. B. Vasil' eva, DAN, **124**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.