

# ON THE REDUCTION OF SYSTEMS OF DIFFERENCE EQUATIONS TO CANONICAL FORM

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1960

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **ON THE REDUCTION OF SYSTEMS OF DIFFERENCE EQUATIONS TO CANONICAL FORM**

*(Presented by Academician S. L. Sobolev on 28 V 1960)*

In the present note we consider a certain class of stable difference systems, investigate the question of reducing these systems to the simplest form, and study the asymptotic behavior of their solutions.

Let the system have the form:

$$u_p^{n+1,k} = \sum_{j,l} a_{a,jl} u_j^{n,k+l}; \quad (1)$$

$p, j = 1, 2, \dots, m$ ;  $l$  is bounded.

We shall call system (1) **stable** <sup>(1)</sup> if

$$\|u_p^n\| \leq K \sum_j \|u_j^0\|, \quad \text{where} \quad \|u_p^n\| = \left\{ \sum_{k=-\infty}^{+\infty} |u_p^{n,k}|^2 \right\}^{1/2}.$$

Introduce the notation

$$U_p^n(s) = \sum_{k=-\infty}^{+\infty} u_p^{n,k} e^{iks} \quad (2)$$

and put in correspondence with system (1) the system

$$U^n = A^n U^0, \quad (3)$$

where  $U^n$  is a vector-function with elements belonging to  $L_2(-\pi, +\pi)$ ;  $A$  is a matrix depending on the parameter  $t = e^{is}$ .

Let the matrix  $A$  be representable in the form

$$A = E + \frac{t-1}{t^\alpha} M(t), \quad (4)$$

where  $M(t)$  is a polynomial matrix which, for  $s = 0$ , has real and distinct eigenvalues (restriction I). At the same time we require that the eigenvalues of  $A(t)$  satisfy the inequality  $|\lambda| \leq q_0 < 1$  for all  $s$  distinct from zero (restriction II).

It is known <sup>(2)</sup> that the eigenvalues of the matrix  $M(t)$  in a neighborhood of any point  $t_0$  are either analytic functions of the parameter  $t$ , or branches of one multivalued analytic function. The number of the latter points is finite, and in a neighborhood of each of them one can choose a number  $k_0$  so that the substitution  $t - t_0 = z^{k_0}$  removes the multivaluedness. Expanding the elements of  $M$  in powers of  $z$ , one can prove the following theorem:

**Theorem.** Suppose there exists a neighborhood of the point  $z = 0$  in which the elements of the matrix  $M(z)$  and its eigenvalues are analytic functions. Then one can indicate, perhaps, a narrower neighborhood of the point  $z = 0$ , in which the matrix  $M(z)$  can be reduced to triangular form by a similarity transformation with a transition matrix nondegenerate in a neighborhood of  $z = 0$ .

In the case when the eigenvalues are distinct,  $M(z)$  can be reduced to diagonal form by a transformation of the indicated type.

By Borel's finite covering lemma, it is possible to divide the interval  $-\pi \leq s \leq \pi$  into a finite number of intervals, in each of which the matrix is reduced to triangular form. Suppose that among the intervals there is one containing the point  $s = 0$  in its interior; we assign it the zero number.

We now represent the matrix  $A(t)$  in the form

$$A(t) = \sum_{i=0}^{k_1} A_i(t), \quad (5)$$

where  $A_i$  is the matrix corresponding to the  $i$ -th interval, coinciding with  $A(t)$  inside it and identically equal to zero outside it.

In this case the matrix  $A^n(t)$  can be represented in the form

$$A^n(t) = \sum_{i=0}^{k_1} A_i^n(t). \quad (6)$$

Each of the matrices  $A_i^n$  can be represented in the form  $A_i^n = C_i^{-1} T_i^n C_i$ , where  $T_i^n$  has the form

$$T_i^n = \begin{pmatrix} \lambda_i^n & & & 0 \\ a_{12} & \lambda_2^n & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ a_{1m} & a_{2m} & \dots & \lambda_m^n \end{pmatrix}, \quad (7)$$

and it can be verified that, by virtue of restriction II, the elements  $a_{ik}$  satisfy the inequalities

$$|a_{ik}| \leq K_1 n^{m-1} q_0^n. \quad (8)$$

Taking equality (6) into account, we rewrite (3) in the form

$$U^n(s) = A_0^n U^0 + \sum_{i=1}^{k_1} A_i^n U^0, \quad (9)$$

where  $A_0 = C_0^{-1} T_0 C_0$ , and  $T_0$  is a diagonal matrix. Hence we obtain that

$$\|u_i^n\| \leq (K_2 + K_1 n^{m-1} q_0^n) \sum_j \|u_j^0\|. \quad (10)$$

From this follows the stability of system (1) and the possibility of replacing it by a simpler one generated by the matrix  $A_0$ , while this new system can be reduced to diagonal form.

Introduce the notation

$$\bar{U}^n = A_0^n U^0. \quad (11)$$

The elements of the matrix  $A_0^n$  are representable in the form

$$a_{p,q}^{(0)} = \sum_{r=1}^m b_{p,q}^r \lambda_r^n. \quad (12)$$

It follows from this that the solution of the difference system corresponding to (11) has the form

$$\bar{u}_j^{k,n} = \frac{1}{2\pi} \sum_{r=1}^m \int_{-\pi}^{+\pi} C_j^r \lambda_r^n e^{-iks} ds, \quad (13)$$

where  $\lambda_r(s)$  are identically zero outside the interval where the matrix  $A_0$  operates. In this case the estimate

$$\sum_{k=-\infty}^{+\infty} |u_j^{n,k} - \bar{u}_j^{n,k}|^2 \leq K_3 n^{m-1} q_0^n$$

holds.

Using asymptotic estimates, one can replace (13) by the more concrete expression

$$\bar{u}_j^{n,k} = \frac{1}{2\pi} \sum_{r=1}^m \int_{-\pi}^{+\pi} \exp \left[ in \sum_{l=1}^{2p_r-1} \beta_l^r s^l - n \beta_{2p_r}^r s^{2p_r} \right] C_j^r e^{-iks} ds. \quad (14)$$

In this case

$$\sum_{k=-\infty}^{+\infty} |u_j^{n,k} - \bar{u}_j^{n,k}|^2 \leq K_4 n^{-3/2\bar{p}+\delta},$$

where  $\bar{p} = \max_r p_r$ , and  $\delta$  is some sufficiently small number.

Each term in (14) is a wave traveling with a certain velocity and smeared over a certain number of points. We note that the velocity of motion of the waves and their smearing for large  $n$  can be judged from the values of the derivatives of the functions  $\ln \lambda_r^n C_j^r$  at the point  $s = 0$ .

Suppose that a function  $v_k$ , belonging to  $l_2$ , is given on the grid; then, to characterize the location of the wave, it is natural to introduce the quantity

$$x = \frac{\sum_{k=-\infty}^{+\infty} k v_k}{\sum_{k=-\infty}^{+\infty} v_k}, \quad (15)$$

and to define the smearing quantity as

$$D_p = R \left[ \left[ \frac{\sum_{k=-\infty}^{+\infty} (k-x)^p v_k}{\sum_{k=-\infty}^{+\infty} v_k} \right] \right]^{1/p}, \quad (16)$$

where  $D_p$  is the first nonzero number in the sequence  $D_2, D_3, \dots$ ;  $R$  is a proportionality coefficient subject to experimental determination. It may happen that infinite series of the form  $\sum_{k=-\infty}^{+\infty} k^\alpha v_k$  diverge; however, certain values can be assigned to their sums<sup>(3)</sup>.

Thus, to the wave corresponding to the eigenvalue  $\lambda_r^n$  and the function  $u_j^{n,k}$ , we assign the numbers

$$x = i[\ln \lambda_r^n C_j^r]_{s=0}'; \quad (17)$$

$$D_p = R \sqrt[p]{\left| \left[ \ln \lambda_r^n C_j^r \right]_{s=0}^{(p)} \right|}. \quad (18)$$

Here it should be required that the initial data allow the existence of derivatives of the proper order for  $C_j^r$ .

We have numerically calculated the following system:

$$\begin{aligned} u_1^{n+1,k} &= u_1^{n,k}(1-r) + u_1^{n,k+1}r + 0.001 \left( u_2^{n,k+1} - 2u_2^{n,k} + u_2^{n,k-1} \right), \\ u_2^{n+1,k} &= u_2^{n,k} + \frac{r}{2} \left( u_2^{n,k+1} - u_2^{n,k-1} \right) + \frac{r^2}{2} \left( u_2^{n,k+1} - 2u_2^{n,k} + u_2^{n,k-1} \right). \end{aligned}$$

Its initial data have the form

$$u_1^{0,k} = \begin{cases} 0, & k \leq 0, \\ 1, & k \geq 1; \end{cases} \quad u_2^{0,k} = \begin{cases} 0, & k \leq 0, \\ 1, & k \geq 1. \end{cases}$$

The present system is stable for  $0 < r < 1$  and can be reduced to diagonal form. According to formulas (17) and (18), we predicted the following values:

$$\begin{aligned} x_1 &= nr + \frac{1}{6}, & D_2^{(1)} &= R_1 \sqrt{\left| n \frac{r(1-r)}{2} - \frac{1}{36} \right|}, \\ x_2 &= nr + \frac{1}{6}, & D_3^{(2)} &= R_2 \sqrt[3]{\left| n \frac{r(1-r^2)}{2} - \frac{1}{135} \right|}, \end{aligned}$$

where  $i_{\frac{1}{6}}$ ,  $i_{\frac{1}{36}}$ , and  $i_{\frac{1}{135}}$  are the values that should be assigned to  $(\ln U^0)'$ ,  $(\ln U^0)''$ , and  $(\ln U^0)'''$  at  $s = 0$  for the initial data chosen by us.

As the computations performed for  $n_1 = 100$ ,  $n_2 = 200$ ,  $r = 0.5$  showed, the computed  $x$  and the experimental  $\bar{x}$  are equal to

$$x_1 = 100.17, \quad \bar{x}_1 = 100.5; \quad x_2 = 100.17, \quad \bar{x}_2 = 99.5.$$

The computation for  $n_1 = 100$  made it possible to determine the proportionality coefficients  $R_1$  and  $R_2$ , after which the computed  $D$  and the experimental  $\bar{D}$  for  $n_2 = 200$  had the values

$$D_2^{(1)} = 16.98, \quad \bar{D}_2^{(1)} = 17; \quad D_3^{(2)} = 6.68; \quad \bar{D}_3^{(2)} = 6.7,$$

which shows satisfactory agreement between theory and experiment.

Received  
17 V 1960

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