MATHEMATICS

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1960

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Abstract

Full Text

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ON A CLASS OF EXTREMAL PROBLEMS FOR POLYNOMIALS

(Presented by Academician S. N. Bernstein on 20 X 1959)

Let E be a locally convex linear topological space (real or complex); $p(x)$ a continuous symmetric convex functional in E; E_n an n-dimensional linear space with a locally convex topology, consisting of points $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, real or complex; $p_1(\lambda) = p_1(\lambda_1, \ldots, \lambda_n)$ a continuous symmetric convex functional in E_n. Let $y, x_1, \ldots, x_n$ be linearly independent elements of E. We shall call linear combinations $\sum_{\nu=1}^{n} \lambda_\nu x_\nu$ **polynomials**. We shall be interested in the following two problems.

Problem $\mathbf{I}_{[p,p_1]}$. Find

$$\alpha = \inf_{\lambda_1,\ldots,\lambda_n} \left[p\left(y - \sum_{\nu=1}^{n} \lambda_\nu x_\nu \right) + p_1(\lambda_1, \ldots, \lambda_n) \right]. \tag{1}$$

Problem II. Find

$$\beta = \sup |f(y)| \tag{2}$$

over all continuous linear functionals $f \in E^*$ satisfying the conditions

$$|f(x)| \le p(x), \quad x \in E; \tag{3}$$

$$\left| \sum_{1} \lambda_\nu f(x_\nu) \right| \le p_1(\lambda_1, \ldots, \lambda_n) \tag{4}$$

for all $(\lambda) \in E_n$.

The close connection between Problems I and II is contained in the following main theorem:

Theorem 1. $\alpha = \beta$.

The proof is based on duality lemmas of functional analysis, which slightly generalize analogous lemmas previously used in investigations of extremal problems ([1–4]), etc. For lack of space we do not formulate these propositions.

Let us give some concrete examples of our problems.

Problem $\mathbf{I}_{[C(Q),p_1]}$. Find

$$\alpha = \inf_{(\lambda)} \left[\max_{t\in Q} \left| y(t) - \sum_1^n \lambda_\nu x_\nu(t) \right| + p_1(\lambda_1, \dots, \lambda_n) \right]; \tag{5}$$

Q is some compact set; $y(t), x_1(t), \dots, x_n(t)$ are continuous functions.

The problem dual to it:

Problem $\mathbf{II}_{[C(Q),p_1]}$. Find

$$\beta = \sup \left| \int_Q y(t)\, dg \right| \tag{6}$$

over all measures g on Q satisfying the inequalities

$$\int_Q |dg| \leqslant 1 \tag{7}$$

and (4) (here $f(x_i) = \int_Q x_i(t)\, dg$).

Further:

Problem $\mathrm{I}_{[L^p,\rho_1]}$, $p \geqslant 1$. Find

$$\alpha = \inf_{(\lambda)} \left[\left\{ \int_a^b \left| y(t) - \sum_1^n \lambda_\nu x_\nu(t) \right|^p dt \right\}^{1/p} + \rho_1(\lambda_1, \dots, \lambda_n) \right]. \tag{8}$$

Problem $\mathrm{II}_{[L^q,\rho_1]}$, $q > 1$. Find

$$\beta = \sup_{\alpha(t)} \left| \int_a^b \alpha(t) y(t)\, dt \right| \tag{9}$$

over all $\alpha(t)$ for which

$$\int_a^b |\alpha(t)|^q dt \leqslant 1 \tag{10}$$

and (4) holds $\left(f(x_i) = \int_a^b \alpha(t) x_i(t)\, dt\right)$. In the case $q = \infty$, inequality (10) becomes

$$|\alpha(t)| \leqslant 1 \quad \text{almost everywhere on } [a, b]. \tag{11}$$

The problems $\mathrm{I}_{[L^p, \rho_1]}$ and $\mathrm{II}_{[L^q, \rho_1]}$ are dual if $\frac{1}{p} + \frac{1}{q} = 1$. Specifying now ρ_1, we obtain, for example, the problem:

Problem $\mathrm{I}_{[C(Q), \{\varepsilon_\nu\}]}$. Find

$$\inf_{(\lambda)} \left[\max_{t \in Q} \left| y(t) - \sum_1^n \lambda_\nu x_\nu(t) \right| + \sum_1^n \varepsilon_\nu |\lambda_\nu| \right],$$

where $\varepsilon_\nu \geqslant 0$ $(\nu = 1, \dots, n)$ are given.

Dual to it is

Problem $\mathrm{II}_{[C(Q), \{\varepsilon_\nu\}]}$. Find (6) under conditions (7) and

$$\left| \int_Q x_\nu(t)\, dg \right| \leqslant \varepsilon_\nu, \quad \nu = 1, \dots, n. \tag{12}$$

In an analogous way we obtain the problems $\mathrm{I}_{[L^p, \{\varepsilon_\nu\}]}$ and $\mathrm{II}_{[L^q, \{\varepsilon_\nu\}]}$. As ρ_1 one may take various other functionals, for example,

$$\left\{ \sum_1^n \varepsilon_\nu |\lambda_\nu|^r \right\}^{1/r}, \quad r \geqslant 1,$$

where $\varepsilon_\nu \geqslant 0$, $\nu = 1, \dots, n$, are given. Or

$$\rho_1(\lambda_1, \dots, \lambda_n) = \left\{ \int_a^b \left| \sum_1^n \lambda_\nu x_\nu(t) \right|^r dt \right\}^{1/r}$$

and so on.

Theorem 2. There exist extremal polynomials

$$P^* = \sum_1^n \lambda_\nu^* x_\nu$$

in problem I and extremal functionals in problem II. In order that the polynomial P^* be extremal in problem I, and the functional f^*, satisfying-

for (3) and (4) to be extremal in Problem II, it is necessary and sufficient that

$$f^*(y - P^*) = e^{i\theta} p(y - P^*); \tag{13}$$

$$f^*(P^*) = \sum_1^n \lambda_\nu^* f^*(x_\nu) = e^{i\theta} p_1(\lambda_1^*, \dots, \lambda_n^*); \tag{14}$$

θ is a real number.

Transferring in a natural way to our case of linear topological spaces the terminology used by M. G. Krein in ([1]), we obtain the following criteria for uniqueness of solutions, analogous to those given in ([1]) for the problems considered there:

Theorem 3. If among the solutions of Problem I there exists at least one solution P^* such that $y - P^*$ is a normal element, then the solution of Problem II is unique up to a constant factor K, $|K| = 1$. If among the solutions of Problem II there is at least one normal one, then the solution of Problem I is unique. In particular, if E is a strictly normed space with norm $p(x)$, then the solution of Problem I is unique.

Simple examples show that strict normedness of the functional $p_1(\lambda)$ does not ensure uniqueness.

We shall denote by B the convex body in the Euclidean space R_n formed by the points $(f(x_1), \dots, f(x_n))$, where $f(x)$ satisfies (3). The following deserves to be noted.

Theorem 4. If E is a strictly normed space with norm $p(x)$, and $f^*(x)$ is a functional extremal in Problem II, then the point $(f^*(x_2), \dots, \dots, f^*(x_n))$ is necessarily an interior point for B. (If E is not strictly normed, then the point $(f^*(x_1, \dots, f^*(x_n))$ may also lie on the boundary of B.)

Consider the problems $\mathrm{I}_{[C(Q),p_1]}$ and $\mathrm{II}_{[C(Q),p_1]}$.

Theorem 5. There exist subsets $Q_r \subset Q$, consisting of r points ($r \leq n+1$ in the real case and $r \leq 2n+1$ in the complex case), such that the solutions of the problems $\mathrm{I}_{[C(Q_r),p_1]}$, $\mathrm{II}_{[C(Q_r),p_1]}$ coincide with the solutions of the problems $\mathrm{I}_{[C(Q),p_1]}$ and $\mathrm{II}_{[C(Q),p_1]}$, respectively.

In the case $p_1 \equiv 0$ we obtain a well-known theorem of approximation theory (([5]-[7]), see also ([8])).

Theorem 6. In order that the polynomial

$$P^*(t) = \sum_1^n \lambda_\nu^* x_\nu(t)$$

be extremal in the problem $\mathrm{I}_{[C(Q),p_1]}$, it is necessary and sufficient that there exist r points $t_1, \dots, t_r \subset Q$ ($r \leq n+1$ in the real case and $r \leq 2n+1$ in the complex case), positive numbers $\mu_1, \dots, \mu_r$, and real numbers $\theta_1, \dots, \theta_r$ such that the relations

$$y(t_j) - P^*(t_j) = Me^{-i\theta_j}, \qquad M = \max_{t \in Q} |y(t) - P^*(t)|, \qquad j = 1, \dots, r; \quad (15)$$

$$\sum_{j=1}^{r} \mu_j e^{i\theta_j} P^*(t_j) = p_1(\lambda_1^*, \dots, \lambda_n^*), \quad (16)$$

hold, while for any polynomial

$$P(t) = \sum_1^n \lambda_\nu x_\nu(t)$$

$$\left| \sum_{j=1}^{n} \mu_j e^{i\theta_j} P(t_j) \right| \leq p_1(\lambda_1, \dots, \lambda_n). \quad (17)$$

Finally,

$$\sum_1^n \mu_j = 1. \quad (18)$$

For $p_1(\lambda_1, \dots, \lambda_n) \equiv 0$, Theorem 6 gives, in the complex case, the theorem of E. Ya. Remez ([9,10]) (see also ([11])), which is another form of the earlier theorem found by A. N. Kolmogorov ([12]), and in the real case the classical theorem of P. L. Chebyshev follows from it.

Theorem 7. *If $x_1(t), \dots, x_n(t)$ is a P. L. Chebyshev system on Q, then for an arbitrary functional $p_1(\lambda_1, \dots, \lambda_n)$ there exist numbers $0 < \delta_1 \leqslant \delta_0$ such that for all $\delta \leqslant \delta_0$ in problems I and $\mathrm{II}_{[C(Q),\delta p_1]}$, for an arbitrary continuous $y(t)$ on Q, for the number of points r in Theorems 5 and 6 we have: $r = n+1$ in the real case, $n+1 \leqslant r \leqslant 2n+1$ in the complex case; for all $\delta \leqslant \delta_1$, for arbitrary $y(t)$ continuous on Q, the solution of problem $\mathrm{I}_{[C(Q),p_1]}$ is unique.*

We note that, if the smallness of δ is not required, both assertions of the theorem cease to be valid.

Theorem 8. *Let $x_1(t), \dots, x_n(t)$ be a real P. L. Chebyshev system. There exists a number δ_2, $0 < \delta_2 \leqslant \delta_1$, such that for all $\delta \leqslant \delta_2$ the unique extremal polynomial in the problem $\mathrm{I}_{[C(Q),\delta p_1]}$ with arbitrary continuous $y(t)$ will be the polynomial P^* of least deviation from $y(t)$ on Q.*

For the complex case Theorem 8 is valid under one additional condition on the system $x_1(t), \dots, x_n(t)$.

Theorem 8 makes it possible to solve effectively the problem $\mathrm{II}_{[C(Q),\{\varepsilon_\nu\}]}$ for small ε_ν (this smallness is expressed by the requirement that certain determinants be positive). For example, the following is solved effectively for small ε_ν.

Problem. Find

$$\sup_g \left| \int_{-1}^{+1} t^n \, dg \right|,$$

if

$$\int_{-1}^{+1} |dg| \leqslant 1, \qquad \left| \int_{-1}^{+1} t^\nu \, dg \right| \leqslant \varepsilon_\nu, \qquad \nu = 1, \dots, n-1.$$

We do not write out the explicit formulas for lack of space.

For problems I and $\mathrm{II}_{[L^p,p_1]}$, Theorem 2 leads to criteria for an extremal polynomial which generalize the known criteria for the polynomial of best approximation in the L^p metrics. In the case $p > 1$, uniqueness of the solution follows from Theorem 3. In the case $p = 1$, uniqueness does not always hold.

Theorem 9. *If $x_1(t), \dots, x_n(t)$ is a P. L. Chebyshev system on (a, b), the function $y(t)$ is continuous on (a, b), and $p_1(\lambda_1, \dots, \lambda_n)$ is such that from the fulfillment of (11) and (4) (here*

$$f(x_\nu) = \int_a^b \alpha(t) x_\nu(t) \, dt$$

) it follows that the point $(f(x_1), \dots, f(x_n))$ belongs to the interior of the body B (see Theorem 4), then the extremal polynomial P^* for the problem $\mathrm{I}_{[L^1,p_1]}$ is unique. In this case the difference $y(t) - P^*(t)$ changes sign at no fewer than n points.

Received
29 IX 1959

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Note: Figure translations are in progress. See original paper for figures.