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Abstract

Full Text

Mathematics

Yu. G. Reshetnyak

Integration over a Convex Polyhedron and Some Questions in the Theory of Linear Inequalities

(Presented by Academician V. I. Smirnov on 27 XI 1959)

Let a system of linear inequalities be given:

$$f_j(X) = \sum_{k=1}^{n} a_{jk}x_k + b_j \geqslant 0 \qquad j = 1, 2, \dots, m. \tag{1}$$

It is assumed that in each row and each column of the matrix $\|a_{jk}\|$ at least one element is nonzero. Let M be the set of points $X(x_1, x_2, \dots, x_n)$—solutions of the system (1). We first consider the following basic problem: to compute the integral of the function $\varphi(X) = \exp\{a_1x_1 + a_2x_2 + \dots + a_{nx}n\}$ over the closed region M (a_k are arbitrary complex numbers).

As is known ([1]), for $l > 0$

$$\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty} \frac{e^{x\zeta}}{\zeta}\,d\zeta = \begin{cases} 1 & \text{for } x > 0, \\ 1/2 & \text{for } x = 0, \\ 0 & \text{for } x < 0. \end{cases} \tag{2}$$

Putting $x = f_j(X)$ in the left-hand side, we obtain the function $\chi_j(X)$; $\chi_j(X) = 1$ for $f_j(X) > 0$ and $\chi_j(X) = 0$ for $f_j(X) < 0$. The product $\chi(X) = \chi_1(X)\cdots\chi_m(X)$ is equal to unity for interior points of the region M and to zero for $X \notin M$. It can be represented in the form of a multiple integral

$$\chi(X) = \frac{1}{(2\pi i)^m}\int_{l_m-i\infty}^{l_m+i\infty}\cdots\int_{l_1-i\infty}^{l_1+i\infty} \frac{\exp\left\{\sum_{j=1}^{m}\left(\sum_{k=1}^{n} a_{jk}x_k + b_j\right)\zeta_j\right\}}{\zeta_1\cdots\zeta_m}\,d\zeta_1\cdots d\zeta_m. \tag{3}$$

The integral here is understood as an iterated one, i.e., the value of the integral is taken to be the result of successive integrations first with respect to ζ_1, then with respect to ζ_2, etc.

We integrate $\varphi(X)\chi(X)$ over the region $Q\{-A_k \leqslant x_k \leqslant R_k\}$, $k = 1, 2, \dots, n$. Changing the order of integration, we obtain:

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$$\int_Q \chi(X)\varphi(X)\,dX = \frac{1}{(2\pi i)^m}\int\cdots\int \frac{e^{\beta(\zeta)}\prod_{k=1}^n \left[e^{B_k\alpha_k(\zeta)} - e^{-A_k\alpha_k(\zeta)}\right]}{\zeta_1\cdots\zeta_m \prod_{k=1}^n \alpha_k(\zeta)}\,d\zeta_1\cdots d\zeta_m, \tag{4}$$

where

$$\beta(\zeta) = \sum_{j=1}^m b_j\zeta_j, \qquad \alpha_k(\zeta) = \sum_{j=1}^m a_{jk}\zeta_j + a_k$$

(the limits of integration in the complex integral are omitted for simplicity).

The right-hand side of (4) can be considerably simplified by sending some of the A_k and B_k to infinity. The real part S_k of the function $a_k(\zeta)$ is equal to $a_{1k}l_1 + \cdots + a_{mk}l_m + \operatorname{Re} a_k$. Let the l_i be chosen so that all $S_k \neq 0$. If $S_k > 0$, then as $A_k \to \infty$, $\exp\{-A_k\alpha_k(\zeta)\} \to 0$. If $S_k < 0$, then as $B_k \to \infty$, $\exp\{B_k\alpha_k(\zeta)\} \to 0$. Passing to the limit, we obtain the integral

$$\frac{1}{(2\pi i)^m}\int\cdots\int_{\zeta_1\cdots\zeta_m} \frac{\pm\exp\gamma(\zeta)}{\prod_{k=1}^n \alpha_k(\zeta)}\,d\zeta_1\ldots d\zeta_m, \tag{5}$$

where $\gamma(\zeta)$ is some linear form in the variables $\zeta_1,\ldots,\zeta_m$. If R_k are those of the A_k and B_k which were fixed in the limiting process just described, then (5) is equal to the integral of the function $\varphi(X)$ over the intersection of the domain M with the n-dimensional "octant" $\varepsilon_k x_k \leq R_k,\ k = 1, 2, \ldots, n$, where $\varepsilon_k = \pm 1$. If the integral (5) is evaluated and then the limit is taken as $R_k \to \infty$, we obtain the value of the integral of $\varphi(X)$ over all of M.

The complex integrals used are, generally speaking, not absolutely convergent, and the justification of the legality of the operations performed requires additional considerations, which we do not give.

The evaluation of the integral (5) is based on the following observation. Let $f(\zeta)$ be a rational function of degree not exceeding -1; let r_1 and r_2 be the sums of the residues of the function $f(\zeta)$ at its singular points lying, respectively, to the left and to the right of the line $\operatorname{Re}\zeta = l$, which contains no singular points of $f(\zeta)$. Then:

$$\frac{1}{2\pi i}\int_{\operatorname{Re}\zeta = l} e^{x\zeta} f(\zeta)\,d\zeta = \begin{cases} r_1, & \text{for } x > 0, \\ \frac{1}{2}(r_1 - r_2), & \text{for } x = 0, \\ r_2, & \text{for } x < 0. \end{cases} \tag{6}$$

(For $x = 0$ one should take the principal value of the integral.)

The integrand in (5), as a function of ζ_1, has singular points:

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$$\zeta_1^{(0)} = 0 \quad \text{for } a_{1k} \neq 0, \qquad \zeta_1^{(k)} = -\frac{1}{a_{1k}} \left(\sum_{j=2}^{m} a_{jk}\zeta_j + a_k \right).$$

Since the real parts of the variables $\zeta_2, \dots, \zeta_m$ are constant, the point ζ_1 always lies on one side of the line $\operatorname{Re}\zeta_1 = l$, and the result of integration with respect to ζ_1 on the basis of (6) is a sum of a finite number of functions of the variables $\zeta_2, \dots, \zeta_m$ of the same form as the original function of the variables $\zeta_1, \zeta_2, \dots, \zeta_m$. Integration of each separate term of the resulting sum with respect to ζ_2 requires the same operations as integration with respect to ζ_1. Thus, the evaluation of the integral (5) requires a finite number of arithmetic operations. In the computation it is convenient to take $R_k = \alpha_{kR},\ k = 1, 2, \dots, n$, where $\alpha_k > 0$, and to regard the number R as sufficiently large. The numbers l_j, without fixing their values, may be regarded as chosen so that all expressions of the form $a_1 l_1 + \dots + a_{ml} m + a$ which can occur in the course of the computation have the sign of the first coefficient different from zero if $a = 0$, and the sign of a if $a \neq 0$. For this it is enough to require that the l_j be sufficiently small and that l_1 be considerably larger than l_2, l_2 considerably larger than l_3, and so on.

Let us consider some applications of the algorithm set forth to the theory of linear inequalities.

Introduce the system of inequalities:

$$\sum_{k=1}^{n} a_{jk} x_k + b_j + \delta_j > 0, \qquad j = 1, 2, \dots, m, \tag{7}$$

where $\delta_j > 0$. Integrating the function $\varphi(X) \equiv 1$, we find, by the preceding, the volume $V = V(R, \delta_1, \dots, \delta_m)$ of the common part M_δ of the solutions of system (7) and the "octant"$\varepsilon_k x_k \le \alpha_k R,\ k = 1, 2, \dots, n$. If R is sufficiently large and the δ_j are small, with δ_1 considerably larger than δ_2, δ_2 considerably larger than δ_3, and so on, then V is a polynomial in R and the δ_j. The degree of the polynomial V with respect to R is equal to the dimension of the limiting cone of the region M of solutions of system (1). If r is the least of the degrees of the monomials with nonzero coefficients constituting V, then $n - r$ is equal to the dimension of the region M. Differentiating V with respect to δ_j, one can determine the $(n-1)$-dimensional area of that face of the region M_δ which corresponds to the j-th inequality of system (7). From this one can determine the dimension of the corresponding face of the region M, which makes it possible to single out from (1) an inequality that is a consequence of the remaining ones.

Suppose it is required to find a supporting plane of the region M, parallel to the plane $f(X) \equiv a_1 x_1 + \dots + a_{nx} n = 0$ (the region M is assumed bounded). If $t = \min_{X \in M} f(X)$, $T = \max_{X \in M} f(X)$, then the required planes have equations $f(X) - t = 0$ and $f(X) - T = 0$. As is easy to show,

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$$T = \lim_{H \to \infty} \log \left(\int_M \exp[Hf(X)] \, dX \right). \tag{8}$$

The integral of the function $\exp[Hf(X)]$ over the region M is equal to a sum of expressions of the form $f(H)e^{Hd}$, where $f(H)$ is a rational function of H. On the basis of (8), for sufficiently large H the greatest of the coefficients d is equal to T.

The minimum t is determined analogously by replacing $f(X)$ by $-f(X)$.

Mathematical Institute
Siberian Branch
Academy of Sciences of the USSR

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1. V. I. Smirnov, *A Course of Higher Mathematics*, 3, part 2, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.

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