ON THE APPLICATION OF PROPERTIES OF DOMAINS OF HOLOMORPHY TO THE STUDY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract

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ON THE APPLICATION OF PROPERTIES OF DOMAINS OF HOLOMORPHY TO THE STUDY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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In this note a new approach is outlined to the study of questions connected with the correctness of problems for one class of differential equations with constant coefficients and even for equations of a more general nature—convolution equations. The methods proposed here have arisen in recent years in connection with the justification of dispersion relations. These methods are based on properties of envelopes of holomorphy for domains of a special type that arise under analytic continuation of (generalized) functions whose properties reflect the basic requirements of quantum field theory.

We shall use the definitions and notation from our preceding note ([1]). By a **generalized function** we mean any linear continuous functional on Schwartz' s space S ([2], [3]). In the space S^* consider the convolution equation

$$u(p) * f_0(p) = f(p), \qquad f \in S^*, \qquad f_0 \in \theta'_c, \tag{1}$$

under the assumption that the given (generalized) functions f and f_0 satisfy the conditions

$$\tilde{f}(x) = 0, \qquad \tilde{f}_0(x) \neq 0 \quad \text{for} \quad x^2 = x_0^2 - \mathbf{x}^2 < 0. \tag{2}$$

Putting

$$f_0(p) = P\left(\frac{\partial}{\partial p_0}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}\right) \delta(p),$$

we conclude that among equations (1) there are contained linear differential equations with constant coefficients

$$P\left(\frac{\partial}{\partial p_0}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}\right) u(p) = f(p) \tag{3}$$

provided that*

$$P(-ix_0, i\mathbf{x}) \neq 0 \quad \text{for} \quad x^2 < 0. \tag{4}$$

We give several examples of such operators P ($m \geq 1$ an integer, $\lambda \geq 0$):

$$\left(\pm\frac{\partial^2}{\partial p_0^2} + \frac{\partial^2}{\partial p_1^2} + \cdots + \frac{\partial^2}{\partial p_n^2} - \lambda\right)^m \quad \text{—elliptic and hyperbolic,}$$

$$\left(\pm\frac{\partial}{\partial p_0} + \frac{\partial^2}{\partial p_1^2} + \cdots + \frac{\partial^2}{\partial p_n^2} - \lambda\right)^m \quad \text{—parabolic.} \tag{5}$$

* The minus sign before ix_0 is due to the fact that the Fourier transform is constructed on the basis of the form $px = p_0x_0 - \mathbf{px}$,

$$\tilde{f}(x) = \int f(p)e^{-ipx}\, dp, \qquad f(p) = \frac{1}{(2\pi)^{n+1}} \int \tilde{f}(x)e^{ipx}\, dx.$$

Passing in equation (1) to the Fourier transform, we obtain

$$\tilde{u}(x)\tilde{f}_0(x) = \tilde{f}(x). \tag{6}$$

The function $\tilde{f}_0 \in \theta_M$ (see ([2]), p. 99). If $\tilde{f}_0(x)$ is an analytic function (in particular, a polynomial), then equation (3), and hence also equation (1), always has a solution in S^* ([5,6]).

It follows from (2) and (6) that every solution of equation (1) from the space S^* is a commutator, i.e. $\tilde{u}(x) = 0$ for $x^2 < 0$. From the results of ([1]) it follows that this solution can be represented as the difference of the boundary values of two functions $F^{\pm}(\zeta)$, holomorphic in $T^{\pm}$, of the classes $N^{\pm}$, respectively:

$$u(p) = \lim_{\varepsilon\to+0} \left[F^+(p_0 + i\varepsilon, \mathbf{p}) - F^-(p_0 - i\varepsilon, \mathbf{p})\right], \tag{7}$$

where the convergence to the limit takes place in the space S^*.

Suppose now that the solution $u(p)$ vanishes in some domain G. Then ([1]) the functions $F^{\pm}(\zeta) = F(\zeta)$ coincide, are holomorphic in $H(T \cup \tilde{G})$, and belong to the class N. In view of (7), the solution $u(p)$ can now be represented in the form

$$u(p) = F(p_0 + i0, \mathbf{p}) - F(p_0 - i0, \mathbf{p}). \tag{8}$$

Thus, the following holds.

Theorem 1. *Every solution from S^* of equation (1) that vanishes in an arbitrary domain G can be represented as the difference (8) of the boundary values of a function $F(\zeta)$ of class N, holomorphic in the domain $H(T \cup \tilde{G})$. This solution, therefore, vanishes in the broader domain $B(G) = \operatorname{Re} H(T \cup \tilde{G})$, consisting of those and only those points p for which every hyperboloid $(p' - u)^2 = s,\ s \geqslant 0$, passing through p, has at least one common interior point with the domain G.*

Corollary 1. Under the conditions of the theorem, the solution is equal to zero in the smallest convex envelope $B_0(G)$ of the domain G with respect to timelike curves.

Corollary 2. If two solutions of equation (1) from S^* coincide in the domain G, then they coincide also in the envelope $B(G) \supset B_0(G)$.

Corollary 3. If two solutions of equation (1) from S^* coincide outside any compact set (or in the domains $p_0^2 < \theta \mathbf{p}^2,\ \theta > 1;\ p_0^2 + m > \mathbf{p}^2,\ m > 0;\ |p_0| < a + |\mathbf{p}|,\ a > 0$, etc.), then they are identical.

Solutions of equation (1) from S^* behave like analytic functions, although they themselves, generally speaking, are not such. Their values inside any infinitely thin domain uniquely determine the behavior of the function in a substantially broader domain. From the results presented it follows that this phenomenon is closely connected with the property of pseudoconvexity of envelopes of holomorphy. For ultrahyperbolic equations these facts were noted in ([6]), Ch. VI, § 8. Similar phenomena were studied by John ([7]) and Nirenberg ([8]) in connection with questions of uniqueness of the solution of the Cauchy problem for general differential equations. Nirenberg ([8]) proposed the hypothesis that if a sufficiently smooth solution of a differential equation with constant coefficients in the principal part vanishes in some domain, then it also vanishes in its smallest convex envelope with respect to timelike curves (here timelike curves are determined by the principal part of the differential operator). From Corollary 1 follows the validity of Nirenberg' s hypothesis for those differential equations (3) for which the timelike curves in both* senses coincide, for example, under addit–

* Our definition of timelike curves is connected with the wave operator.

to the condition (4) $P(-ix, ix) = 0$ for $x^2 = 0$. The method of proof used here differs from the methods previously applied in the study of similar questions [(6–8)]. It would be interesting to generalize this method to ultrahyperbolic or strictly hyperbolic differential equations with constant coefficients.

It follows from Theorem 1 that, for differential equations (3), the initial Cauchy data cannot be prescribed arbitrarily on surfaces of a spatially similar type. More precisely, for the equation

$$\frac{\partial^k u}{\partial p_n^k} = P_1\left(\frac{\partial}{\partial p_0}, \dots, \frac{\partial}{\partial p_n}\right) u, \tag{9}$$

solved with respect to the highest derivative in p_n and satisfying condition (4), the following is true:

Theorem 2. For there to exist a solution $u \in S^*$ of equation (9), vanishing together with its derivatives with respect to p_n up to order $k-1$ inclusive in a domain g of the hyperplane $p_n = 0$, it is necessary that these zero data vanish also in the envelope $B(g)$ of the domain g, in particular in its least convex envelope $B_0(g)$ with respect to timelike curves (in the hyperplane $p_n = 0$).

Proof. Let the solution u of equation (9) belong to S^* and satisfy the conditions of the theorem. Let, further, $\bar\eta(p_0, \dots, p_{n-1})$ be a finite infinitely differentiable function such that $\bar\eta \geq 0$,

$$\int \bar\eta \, dp_1 \cdots dp_{n-1} = 1.$$

Then

$$\eta_\varepsilon(p_0, \dots, p_{n-1}) \equiv \varepsilon^{-n} \bar\eta\left(\frac{p_0}{\varepsilon}, \dots, \frac{p_{n-1}}{\varepsilon}\right) \to \delta(p_0) \cdots \delta(p_{n-1}) \quad \text{as } \varepsilon \to +0.$$

By virtue of the hypoellipticity of equation (9) in p_n [9], the convolution

$$u_\varepsilon(p) = \int u(p_0', \dots, p_{n-1}', p_n) \, \eta_\varepsilon(p_0 - p_0', \dots, p_{n-1} - p_{n-1}') \, dp_0' \cdots dp_{n-1}'$$

is an infinitely differentiable function of all arguments, satisfies equation (9), and has zero Cauchy initial data in the domain g_ε of the hyperplane $p_n = 0$, where g_ε consists of those points of g which are at a distance from the boundary of g not less than $c\varepsilon$ ($c > 1$). By Holmgren' s theorem [10], p. 49, the function $u_\varepsilon(p)$ vanishes in some $(n+1)$-dimensional neighborhood g_ε' of the domain g_ε. By Theorem 1, $u_\varepsilon(p) = 0$ for $p \in B(g_\varepsilon')$. Since $g_\delta \subset g_\varepsilon \subset g$, it follows that $B(g_\delta') \subset B(g_\varepsilon')$, if $\delta \gg \varepsilon > 0$. Therefore the disk

$$\pi_\delta : \quad \{(p_0, \dots, p_{n-1}) \in g_\delta, \ |p_n| < \delta\}$$

belongs to all neighborhoods g_ε', starting with sufficiently small $\varepsilon < \delta$. Since $u_\varepsilon(p) \to u(p)$ as $\varepsilon \to +0$ (in S^*), it has become clear that $u(p) = 0$ for $p \in \pi_\delta$. In view of the arbitrariness of δ, $u(p) = 0$ in some $(n+1)$-dimensional neighborhood g' of the domain g. By Theorem 1, $u(p) = 0$ for $p \in B(g')$. But the section of the domain $B(g')$ by the hyperplane $p_n = 0$ coincides with $B(g) \supset B_0(g)$. The theorem is proved.

Remark. The operators (5) are hypoelliptic in each variable separately [9].

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Note: Figure translations are in progress. See original paper for figures.