Mathematics

D. F. DAVIDENKO

1960

SovietRxiv

Abstract

Full Text

Mathematics

D. F. DAVIDENKO

ON THE APPLICATION OF THE PARAMETER-VARIATION METHOD TO MATRIX INVERSION

(Presented by Academician N. N. Bogolyubov on 16 XI 1959)

In notes ([1,2]) we proposed a parameter-variation method for the numerical solution of systems of nonlinear algebraic and transcendental equations. This method has also proved applicable to the inversion of matrices.

1°. Let $A(\lambda) = \|a_{ik}(\lambda)\|$ $(i, k = 1, 2, \dots, n)$ be a square matrix of order n, whose elements are functions of a parameter λ taking prescribed values on a finite interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$.

We shall call the matrix $B(\lambda) = \|b_{ik}(\lambda)\|$ $(i, k = 1, 2, \dots, n)$ the inverse of the matrix $A(\lambda)$ if

$$A(\lambda)B(\lambda) = E, \tag{1}$$

where E is the identity matrix.

Suppose that for some value of the parameter λ, for example $\lambda = \lambda_0$, the inverse matrix $B(\lambda_0)$ is known for the matrix $A(\lambda_0)$:

$$B(\lambda_0) = B_0 = \|b^0_{ik}\|, \qquad i, k = 1, 2, \dots, n. \tag{2}$$

Suppose further that the functions $a_{ik}(\lambda)$ $(i, k = 1, 2, \dots, n)$ are defined and continuous on the entire interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$ and have continuous derivatives on this interval. Suppose, moreover, that the matrix $A(\lambda)$ has a nonzero determinant for all λ in the interval under consideration, i.e.

$$\operatorname{Det} A(\lambda) \neq 0, \qquad \lambda_0 \leqslant \lambda \leqslant \lambda^*.$$

It is required to determine the inverse matrix $B(\lambda)$ for all prescribed values of the parameter $\lambda > \lambda_0$.

To this end we proceed as follows. Taking λ as an independent variable and differentiating equality (1) with respect to λ, we have

$$\frac{dA(\lambda)}{d\lambda}B(\lambda) + A(\lambda)\frac{dB(\lambda)}{d\lambda} = 0.$$

Hence

$$\frac{dB(\lambda)}{d\lambda} = -B(\lambda)\frac{dA(\lambda)}{d\lambda}B(\lambda). \tag{3}$$

Equation (3) represents, generally speaking, n^2 ordinary differential equations for the unknown elements of the inverse matrix $B(\lambda)$.

To determine the elements of the matrix $B(\lambda)$ for all prescribed values of λ, we numerically integrate equation (3) by one of the methods of numerical integration of ordinary differential equations (see, for example, ([3,4])) on the interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$ with the initial condition (2):

$$\text{for } \lambda = \lambda_0 \qquad B(\lambda) = \|b_{ik}^0\|, \qquad i, k = 1, 2, \dots, n.$$

In this case the general scheme of computations is characterized by the uniformity and simplicity of the operations performed and is conveniently implemented on modern computing machines.

It is hardly necessary to note that if the matrix $A(\lambda)$ is symmetric, then the amount of computation is greatly reduced, since the elements of the matrix $B(\lambda)$ symmetric with respect to the main diagonal are equal.

The advantage of the proposed method is the circumstance that throughout all computations the operation of division is absent. Thus cases of division by a small number are eliminated, which significantly improves the accuracy of the obtained elements of the inverse matrices. Moreover, in many particular cases the operation of subtraction is also absent. Owing to this, in these cases the possibility of loss of signs is completely eliminated and the reliability of the final result is guaranteed.

Below we give some of these particular cases, which, of course, do not exhaust all cases that may be encountered in practice.

Particular cases. 1. All elements of the matrix $dA(\lambda)/d\lambda$ for all λ from the interval under consideration satisfy the condition

$$a'_{ik}(\lambda) \geqslant 0 \ (a'_{ik}(\lambda) \leqslant 0), \qquad i, k = 1, 2, \dots, n*$$

(the prime denotes differentiation with respect to λ), and the elements of the initial inverse matrix B_0 satisfy the condition

$$b_{ik}^0 \leqslant 0 \ (b_{ik}^0 \geqslant 0), \qquad i, k = 1, 2, \dots, n.$$

2. For some integer $p < n$, all elements b^0_{ik} $(i, k = 1, 2, \dots, n)$ of the matrix B_0 satisfy the conditions

$$b^0_{ik} \leqslant 0 \ (b^0_{ik} \geqslant 0), \qquad i = 1, 2, \dots, p;\ k = 1, 2, \dots, n;$$

$$b^0_{ik} \geqslant 0 \ (b^0_{ik} \leqslant 0), \qquad i = p+1, p+2, \dots, n;\ k = 1, 2, \dots, n,$$

and the elements $a'_{ik}(\lambda)$ $(i, k = 1, 2, \dots, n)$ of the matrix $dA(\lambda)/d\lambda$, for all $\lambda_0 \leqslant \lambda \leqslant \lambda^*$, satisfy the conditions

$$a'_{ik}(\lambda) \geqslant 0 \ (a'_{ik}(\lambda) \leqslant 0), \qquad i = 1, 2, \dots, n;\ k = 1, 2, \dots, p;$$

$$a'_{ik}(\lambda) \leqslant 0 \ (a'_{ik}(\lambda) \geqslant 0), \qquad i = 1, 2, \dots, n;\ k = p+1, p+2, \dots, n.$$

For $p = n$ we have, evidently, case 1.

3. All elements b^0_{ik} $(i, k = 1, 2, \dots, n)$ of the matrix B_0 satisfy the conditions

$$\begin{aligned} b^0_{ik} \geqslant 0 \ (b^0_{ik} \leqslant 0), &\qquad i + k = 2m; \\ b^0_{ik} \leqslant 0 \ (b^0_{ik} \geqslant 0), &\qquad i + k = 2m+1, \end{aligned}$$

and the elements $a'_{ik}(\lambda)$ $(i, k = 1, 2, \dots, n)$ of the matrix $dA(\lambda)/d\lambda$ satisfy the conditions

$$\begin{aligned} a'_{ik}(\lambda) \leqslant 0 \ (a'_{ik}(\lambda) \geqslant 0), &\qquad i + k = 2m; \\ a'_{ik}(\lambda) \geqslant 0 \ (a'_{ik}(\lambda) \leqslant 0), &\qquad i + k = 2m+1, \end{aligned}$$

$$\lambda_0 \leqslant \lambda \leqslant \lambda^*.$$

4. For some $p < n$, all elements b^0_{ik} $(i, k = 1, 2, \dots, n)$ of the initial matrix B_0 satisfy the following conditions:

$$b^0_{ik} \leqslant 0 \ (b^0_{ik} \geqslant 0), \qquad i = 1, 2, \dots, p;\ k = 1, 2, \dots, p;$$

$$b^0_{ik} \geqslant 0 \ (b^0_{ik} \leqslant 0), \qquad i = p+1, p+2, \dots, n;\ k = 1, 2, \dots, p;$$

$$b^0_{ik} \geqslant 0 \ (b^0_{ik} \leqslant 0), \qquad i = 1, 2, \dots, p;\ k = p+1, p+2, \dots, n;$$

$$b^0_{ik} \leqslant 0 \ (b^0_{ik} \geqslant 0), \qquad i = p+1, p+2, \dots, n;\ k = p+1, p+2, \dots, n,$$

and the elements $a'_{ik}(\lambda)$ $(i, k = 1, 2, \dots, n)$ of the matrix $dA(\lambda)/d\lambda$ on the entire interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$ satisfy the conditions

$$a'_{ik}(\lambda) \geqslant 0 \ (a'_{ik}(\lambda) \leqslant 0), \qquad i = 1, 2, \dots, p; \ \ k = 1, 2, \dots, p;$$

* Here and in what follows we shall assume that the equality sign cannot occur simultaneously for all element

$$a'_{ik}(\lambda) \leqslant 0 \ (a'_{ik}(\lambda) \geqslant 0), \quad i = p+1, \ p+2, \dots, n; \ \ k = 1, 2, \dots, p;$$

$$a'_{ik}(\lambda) \leqslant 0 \ (a'_{ik}(\lambda) \geqslant 0), \quad i = 1, 2, \dots, p; \ \ k = p+1, \ p+2, \dots, n;$$

$$a'_{ik}(\lambda) \geqslant 0 \ (a'_{ik}(\lambda) \leqslant 0), \quad i = p+1, \ p+2, \dots, n; \ \ k = p+1, \ p+2, \dots, n.$$

Here also, for $p = n$, we have case 1.

5. For the elements b^0_{ik} $(i, k = 1, 2, \dots, n)$ the following conditions hold:

$$b^0_{ik} \leqslant 0 \ (b^0_{ik} \geqslant 0), \quad i = 2m+1; \ \ k = 1, 2, \dots, n;$$

$$b^0_{ik} \geqslant 0 \ (b^0_{ik} \leqslant 0), \quad i = 2m; \ \ k = 1, 2, \dots, n,$$

and for the elements $a'_{ik}(\lambda)$ $(i, k = 1, 2, \dots, n)$—the conditions

$$a'_{ik}(\lambda) \geqslant 0 \ (a'_{ik}(\lambda) \leqslant 0), \quad i = 1, 2, \dots, n; \ \ k = 2m+1;$$

$$a'_{ik}(\lambda) \leqslant 0 \ (a'_{ik}(\lambda) \geqslant 0), \quad i = 1, 2, \dots, n, \ \ k = 2m;$$

$$\lambda_0 \leqslant \lambda \leqslant \lambda^*.$$

It should be noted that in these special cases the elements of the matrix being determined, $B(\lambda)$, for any λ in the interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$, will have the same signs as the corresponding elements of the matrix B_0.

2°. The proposed method is also applicable to the inversion of constant matrices. Indeed, any constant n-th order matrix C, whose inverse we seek, can be represented as the sum of two matrices C_0 and C_1 in such a way that the matrix C_0^{-1}, inverse to C_0, is easily determined. Then the matrix

$$C_\lambda = C_0 + \lambda C_1$$

for $\lambda = 1$ coincides with the original matrix C, and for $\lambda = 0$ has the inverse C_0^{-1}. Proceeding with the matrix C_λ analogously to the preceding case (1°), we obtain, for determining the elements of the inverse matrix C_λ^{-1}, the matrix equation

$$\frac{dC_\lambda^{-1}}{d\lambda} = -C_\lambda^{-1} C_1 C_\lambda^{-1}. \tag{4}$$

The obtained equation (4) is numerically integrated on the interval $0 \leqslant \lambda \leqslant 1$ under the initial condition

$$\lambda = 0, \quad C_\lambda^{-1} = C_0^{-1}.$$

In this case, the matrix C_λ^{-1} obtained at $\lambda = 1$ will be the desired inverse matrix to the matrix C.

Remark 1. The efficiency of inverting the matrix C depends to a considerable extent on the way it is decomposed into the sum of two matrices C_0 and C_1.

Remark 2. The inversion of an arbitrary matrix C can be reduced to the inversion of a symmetric matrix and two matrix multiplications by means of the formula $C^{-1} = C^*(CC^*)^{-1}$, where C^* is the matrix transposed to the matrix C. The inversion of a symmetric matrix is, as already noted, a considerably simpler problem than the inversion of an arbitrary matrix C.

Received
22 X 1959

CITED LITERATURE

1. D. F. Davidenko, DAN, 88, No. 4 (1953).
2. D. F. Davidenko, *Ukr. matem. zhurn.*, 5, No. 2 (1953).
3. A. N. Krylov, *Lectures on Approximate Calculations*, Moscow-Leningrad, 1950.
4. L. Collatz, *Numerical Methods for Solving Differential Equations*, Moscow, 1953.