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Abstract

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MATHEMATICAL PHYSICS

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ON THE QUESTION OF THE PROOF OF THE DOUBLE SPECTRAL REPRESENTATION

(Presented by Academician N. N. Bogolyubov, 9 XI 1959)

In connection with the appearance of works on double dispersion relations, the question arose of obtaining a spectral representation of generalized functions depending on two variables. At one of the seminars of the IX International Conference on High Energies in Kiev, N. N. Bogolyubov expressed the idea of the possibility of formulating general conditions under which, for a function $f(z_1, z_2)$, the representation

$$\begin{aligned} f(z_1, z_2) = & \int_{-\infty}^{+\infty} dv_1 \int_{-\infty}^{+\infty} dv_2 \, \frac{s_3(v_1, v_2)}{(z_1 - v_1)(z_2 - v_2)} \\ & + \int_{-\infty}^{+\infty} dv_1 \int_{-\infty}^{+\infty} dv_2 \, \frac{s_2(v_1, v_2)}{(z_1 - v_1)(z_3 - v_2)} \\ & + \int_{-\infty}^{+\infty} dv_1 \int_{-\infty}^{+\infty} dv_2 \, \frac{s_1(v_1, v_2)}{(z_2 - v_1)(z_3 - v_2)}, \end{aligned} \tag{1}$$

is admissible, where $z_1 + z_2 + z_3 = \text{const}$. The aim of the present work is to prove this.

Let us consider the space U of all functions $u(x, y)$

$$-\infty < x < +\infty, \qquad -\infty < y < +\infty, \tag{2}$$

possessing the following properties:

1) $u(x, y)$ is continuous, together with u_x, u_y, and u_{xy}, in all quadrants of the plane (2), while on the lines $x = 0$, $y = 0$ discontinuities of the first kind may occur;

2) $x^k y^m u(x, y)$, $x^k y^m u_x(x, y)$, $x^k y^m u_y(x, y)$, $x^k y^m u_{xy}(x, y)$ are absolutely integrable in the plane (2) for any k, m;

3) on the lines $x = 0$, $y = 0$ there is summability of the first partial derivatives and of the mixed derivative of $u(x, y)$, $u(-x, y)$, $u(x, -y)$, and $u(-x, -y)$.

We introduce in U a topology by specifying a countable set of norms:

$$\|u\|_{k,m} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \,\left|x^k y^m u(x,y)\right| + \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \,\left|x^k y^m u_x(x,y)\right| + \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \,\left|u_y(x,y) x^k y^m\right| + \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \,\left|x^k y^m u_{xy}(x,y)\right| + \int_0^{+\infty} dy \,\left|u_y(0,y)\right| + \int_0^{+\infty} dy \,\left|u_y(0,-y)\right| + \int_0^{+\infty} dx \,\left|u_x(x,0)\right| + \int_0^{+\infty} dx \,\left|u_x(-x,0)\right| +$$

$$+ \int_0^{+\infty} dy \,|u_{xy}(0,y)| + \int_0^{+\infty} dy \,|u_{xy}(0,-y)| + \int_0^{+\infty} dx \,|u_{xy}(x,0)| + \int_0^{+\infty} dx \,|u_{xy}(-x,0)| + \max|u| + \max|u_x| + \max|u_y| + \max|u_{xy}|. \quad (3)$$

Then U turns out to be a complete countably normed space. Let us determine the space $\widetilde{U}$, dual to U, of functions $\tilde{u}$ obtained by the Fourier transform. All functions from $\widetilde{U}$ are infinitely differentiable by virtue of the properties of $u(x,y) \in U$. We shall show that $\tilde{u} \in \widetilde{U}$ decrease at infinity no more slowly than $(xy)^{-1}$.

The space U is defined in such a way that it is sufficient to consider

$$\int_0^{\infty} dx' \int_0^{\infty} dy' \, u(x',y') \exp\{i(x'x + y'y)\}. \quad (4)$$

We shall integrate (4) by parts, taking into account that $u(x,y)$ vanishes at infinity:

$$\int_0^{\infty} dx' \int_0^{\infty} dy' \, u(x',y') \exp\{i(x'x + y'y)\} =$$

$$= \frac{1}{ix} \int_0^{\infty} dy' \left[u(x',y') \exp(ix'x) \Big|_0^{\infty} \exp(iy'y) - \int_0^{\infty} dx' \, u_{x'}(x',y) \exp\{i(x'x + y'y)\} \right] =$$

$$= -\frac{1}{xy} \left[u(+0,+0) + \int_0^{\infty} dy' \, u_{y'}(+0,y') \exp(iy'y) \right] -$$

$$-\frac{1}{ix} \int_0^{\infty} dx' \int_0^{\infty} dy' \, u_{x'}(x',y') \exp\{i(x'x + y'y)\}.$$

Carrying out for

$$\frac{1}{x}\int dx'\int dy'\,u_{x'}(x',y')\exp\{i(x'x+y'y)\}$$

the same transformations as for (4), we immediately obtain the required assertion.

In the argument the summability of u_x, u_y, and u_{xy} on the axes was essentially used.

For each function $\tilde{u}$ convolution with the generalized function ([1,2])

$$\frac{1}{x}\frac{1}{y}, \tag{5}$$

belonging to $\widetilde{U}$, is possible; $(xy)^{-1}$ is the generalized function

$$\beta(x,y)=\begin{cases}1, & \text{for } x>0,\ y>0 \text{ or } x<0,\ y<0,\\ -1, & \text{otherwise.}\end{cases} \tag{6}$$

Convolution with $(xy)^{-1}$ in $\widetilde{U}$ passes into the operation of multiplication by β in U. But the latter maps U into itself. Thus, there is the expression

$$f(z_1,z_2)=\int_{-\infty}^{+\infty}dv_1\int_{-\infty}^{+\infty}dv_2\,\frac{\tilde{u}(v_1,v_2)}{(z_1-v_1)(z_2-v_2)}, \tag{7}$$

which is a certain generalized function $f\in\widetilde{U}$. Applying twice to $u(x,y)$ the operation of multiplication by β, we return again to $u(x,y)$. Therefore, applying the convolution operation to f, we arrive at the original function $\tilde{u}$

$$\tilde{u}(v_1,v_2)=\int_{-\infty}^{+\infty}dz_1\int_{-\infty}^{+\infty}dz_2\,\frac{f(z_1,z_2)}{(v_1-z_1)(v_2-z_2)}. \tag{8}$$

The results carry over to generalized functions considered as linear continuous functionals on the space $\widetilde{U}$. For example, if $\Phi(x,y)$ is locally integrable and grows at infinity no faster than $(xy)^{-\varepsilon}$ $(\varepsilon>0)$, then it defines a certain functional on the space $\widetilde{U}$, and formulas (7) and (8) are valid.

To obtain representation (1), it is enough to split $\tilde{u}$ into three parts

$$\tilde{u}(v_1,v_2)=s_1(v_1,v_2)+s_2(v_1,v_2)+s_3(v_1,v_2) \tag{9}$$

and make a simple change of variables, using $z_1+z_2+z_3=\text{const}$.

The theorem is proved.

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REFERENCES

1. E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 1948.

2. I. M. Gelfand, G. E. Shilov, *Spaces of Fundamental and Generalized Functions*, 1958.

Note: Figure translations are in progress. See original paper for figures.