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Abstract

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MATHEMATICS

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DIRECT DECOMPOSITIONS OF IDEMPOTENTS IN SEMIGROUPS

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A. G. Kurosh ([1]) began the construction of a theory of direct decompositions in categories with partial addition of mappings; these categories include, as a rule, those categories of algebraic structures in which it is possible to develop a theory of direct decompositions. The program for the further development of the theory of direct decompositions in categories, outlined by Kurosh, was realized by the author of the present communication in works ([2,3]). Thus there are two parallel theories of direct decompositions: a structure-theoretic one and a category-theoretic one. Naturally there arises the problem, posed by Kurosh ([2]), of unifying these theories; the present paper is devoted to the solution of this problem. This unification takes place in the language of semigroup theory. In a semigroup with zero and identity, essentially maximal systems of pairwise orthogonal idempotents are considered, and the relations between such systems are studied.

Let H be an arbitrary semigroup with zero ω and identity ε. In the semigroup H we single out some fixed subsemigroup P containing ω and ε. The elements of the subsemigroup P will be called P-elements.

Definition 1. An idempotent P-element χ of the subsemigroup H is called a **direct sum** of the elements $\alpha_i,\ i \in I$,

$$\chi = \dot{\sum_{i\in I}} \alpha_i, \tag{1}$$

if for all $i \in I$: 1) $\alpha_i \neq \omega$ are P-elements; 2) $\alpha_i^2 = \alpha_i$; 3) $\chi\alpha_i = \alpha_i\chi = \alpha_i$; 4) $\alpha_i\alpha_j = \omega$ for $i \neq j$; 5) for an arbitrary element $\beta \in H$: a) from $\chi\beta = \beta$ and $\alpha_i\beta = \omega$ for all $i \in I$ it follows that $\beta = \omega$; b) from $\beta\chi = \beta$ and $\beta\alpha_i = \omega$ for all $i \in I$ it follows that $\beta = \omega$. The direct sum (1) is called a **direct decomposition** of the element χ, and $\alpha_i,\ i \in I$, are called **direct summands** of the element χ.

Definition 2. If each direct summand $\alpha_i,\ i \in I$, of (1) is decomposed into a direct sum

$$\alpha_i = \dot{\sum_{j \in J_i}} \alpha_{ij}, \tag{2}$$

then there is a direct decomposition

$$\chi = \dot{\sum_{i \in I,\ j \in J_i}} \alpha_{ij}, \tag{3}$$

which is called a **continuation** of the direct decomposition (1).

Definition 3. Some of the direct decompositions of idempotent P-elements χ of the semigroup H will be called S**-direct decompositions**. In doing so, we shall assume that the following conditions are satisfied:

1) every direct decomposition with a finite number of summands is an S-direct decomposition;
2) if the direct decompositions (1) and (2) are S-direct decompositions, then the direct decomposition (3) is also an S-direct decomposition;
3) if the direct decomposition (3) is an S-direct decomposition, then the direct decompositions (1) and (2) are also S-direct decompositions;
4) if the direct decomposition (1) is an S-direct decomposition and if the P-element $\varphi\chi\psi$, where φ and ψ are P-elements representable as products of S-direct summands of the element χ, is a direct sum of P-elements $\varphi\alpha_i\psi,\ i \in I$,

$$\varphi\chi\psi = \dot{\sum_{i \in I}} \varphi\alpha_i\psi,$$

then this direct decomposition is an S-direct decomposition.

Definition 4. An idempotent P-element χ is called S**-regular** if: 1) for an arbitrary S-direct decomposition (1) and an arbitrary subset T of the set of indices I, there is an idempotent P-element α_T such that: a) $\chi\alpha_T = \alpha_T\chi = \alpha_T$; b) $\alpha_i\alpha_T = \alpha_T\alpha_i = \alpha_i$ for $i \in T$; c) $\alpha_i\alpha_T = \alpha_T\alpha_i = \omega$ for $i \notin T$; 2) for any $\beta \in H$ and an arbitrary direct decomposition $\chi = \alpha_1 + \alpha_2$ with two summands: a) from $\chi\beta = \beta$ and $\alpha_1\beta = \omega$ ($\alpha_2\beta = \omega$) it follows that $\alpha_2\beta = \beta$ ($\alpha_1\beta = \beta$); b) from $\beta\chi = \beta$ and $\beta\alpha_1 = \omega$ ($\beta\alpha_2 = \omega$) it follows that $\beta\alpha_2 = \beta$ ($\beta\alpha_1 = \beta$). If, as the set of S-direct decompositions, the set of all finite direct decompositions is selected, then we shall call an S-regular element **regular**. Clearly, every S-regular element is regular.

Definition 5. A semigroup H is called S-**regular** if its identity ε is an S-regular element and for every idempotent P-element $\alpha \neq \varepsilon$, $\alpha \neq \omega$ there is an idempotent P-element $\bar{\alpha} \neq \omega$, $\bar{\alpha} \neq \varepsilon$ such that $\varepsilon = \alpha + \bar{\alpha}$. If, as the set of S-direct decompositions, the set of all finite direct decompositions is selected, then an S-regular semigroup will be called **regular**. Clearly, every S-regular semigroup is regular.

Definition 6. An element γ of a regular semigroup is called **normal** if, from the fact that the element is representable as a product of γ and arbitrary direct summands of the identity ε, it follows that it is a direct summand of the identity ε.

Definition 7. We shall say that an idempotent P-element χ of a regular semigroup H **satisfies the splitting hypothesis** if, for any direct summand α of the element χ and for any pair of complementary direct summands $\beta, \bar{\beta}$ of the element χ, $\chi = \beta + \bar{\beta}$, the following conditions are fulfilled:

1. $\alpha = \alpha_1 + \alpha_2$.
2. There exist normal elements γ_1 and γ_2 such that

$$\gamma_1 \cdot \alpha\beta\alpha = \alpha\beta\alpha \cdot \gamma_1 = \alpha_1, \qquad \gamma_2 \cdot \alpha\bar{\beta}\alpha = \alpha\bar{\beta}\alpha \cdot \gamma_2 = \alpha_2.$$

3. $\alpha_1\gamma_1 = \gamma_1\alpha_1 = \alpha_1, \quad \alpha_2\gamma_2 = \gamma_2\alpha_2 = \gamma_2.$

Definition 8. Elements α and β of the semigroup H are called **right-associated** if $\alpha\beta = \beta$, $\beta\alpha = \alpha$; elements α and β are called **left-associated** if $\alpha\beta = \alpha$, $\beta\alpha = \beta$. A direct summand α of an idempotent P-element χ is called **subordinate** to the direct summand β of the element χ if there exists a direct summand γ of the element χ such that α and γ are right-associated, and β and γ are left-associated.

Definition 9. We shall say that the S-direct decompositions of an idempotent P-element χ of an S-regular semigroup H

$$\chi = \dot{\sum_{i \in I}} \alpha_i = \dot{\sum_{j \in J}} \beta_j$$

possess S-**special continuations** if there exist such S-direct decompositions

$$\alpha_i = \dot{\sum_{j \in J}} \alpha_{ij}, \quad i \in I; \qquad \beta_j = \dot{\sum_{i \in I}} \beta_{ji}, \quad j \in J,$$

that for any $i \in I$ and for any subset J' of the set J the element $\sum_{j \in J'} \alpha_{ij}$ is subordinate to the element $\sum_{j \in J'} \beta_{ji}$. If the set of finite direct decompositions is chosen as the set of S-direct decompositions in the semigroup H, then the S-special extensions will be called **special**.

Theorem 1. If an idempotent P-element χ of a regular semigroup H satisfies the splitting hypothesis, then any two direct decompositions of the element χ with a finite number of summands each have special extensions.

Theorem 2. Two S-direct decompositions of an idempotent P-element χ of an S-regular semigroup H

$$\chi = \sum_{i\in I} \alpha_i = \sum_{j\in J} \beta_j$$

have a common S-extension if and only if $\alpha_i\beta_j = \beta_j\alpha_i$ for all $i \in I,\ j \in J$.

Definition 10. An idempotent P-element χ of an S-regular semigroup H is called an FS-element if, for any S-direct summand β of the element χ and any pair of complementary direct summands $\alpha,\ \bar{\alpha}$ of the element χ, $\chi = \alpha + \bar{\alpha}$, the condition $\bar{\alpha}\beta\alpha = \omega$ is satisfied.

Theorem 3. If an idempotent P-element χ of an S-regular semigroup H satisfies the splitting hypothesis and from each S-direct decomposition of the element χ one can select a finite number of S-direct summands so that the sum of the remaining S-direct summands is an FS-element, then any two S-direct decompositions of the element χ have S-special extensions.

Without essential changes in comparison with the work ([2]), all notions and results of § 2, nos. 15–19, and § 3 of that work carry over to the case of an idempotent P-element χ of an S-regular semigroup H satisfying the splitting hypothesis. In particular, the following three theorems hold.

Theorem 4. Let an idempotent P-element χ of an S-regular semigroup H satisfy the splitting hypothesis and condition (**). If the idempotent P-element χ of the S-regular semigroup H is decomposed into an S-direct sum

$$\chi = \sum_{i\in I} \alpha_i, \tag{4}$$

then for any indecomposable direct summand β of the element χ one can select a finite number of direct summands of the decomposition (4), $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n}$, in such a way that there is no element $\gamma \in H$ for which

$$\bar{\beta}\alpha\gamma = \beta,$$

where $\alpha = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_n}$.

Then, for any two S-direct decompositions of the element χ with indecomposable summands, to each direct summand of one of the decompositions one can put in correspondence such a direct summand of the other decomposition that the corresponding direct summands are subordinate to one another.

Theorem 5. Let an idempotent P-element χ of an S-regular semigroup H satisfy the splitting hypothesis and condition (**). Then, if two S-direct decompositions of the element χ are given,

$$\chi = \sum_{j \in J} \beta_j = \alpha_1 + \alpha_2 + \cdots + \alpha_i + \cdots \tag{5}$$

with indecomposable summands each time, and if the second decomposition (5) contains a finite or countable number of summands, then the first decomposition (5) contains the same number of summands, and the summands of the first decomposition can be numbered so that the element $\beta_1 + \beta_2 + \cdots + \beta_n$ is subordinate to the element $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ for every n.

Theorem 6. If an idempotent P-element χ of an S-regular semigroup H satisfies the splitting hypothesis and condition ($**$), then any two S-direct decompositions of the element χ with indecomposable summands are similar term by term.

From Theorem 1 follows Theorem 2 of the paper [(3)] and the theorem of Mochulsky [(4)]. From Theorem 2 follow the corresponding theorems of Kurosh [(1)], § 4, item 1, and [(5)]. From Theorems 4, 5, and 6 follow both Theorems 6.6, 6.7, and 6.8 from the author' s paper [(2)], and the corresponding structure-theoretic results of Baer [(6)]. In addition, from Theorem 2 follows the following

Theorem 7. Any two decompositions of an associative ring with identity into a complete direct sum have a common continuation.

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Note: Figure translations are in progress. See original paper for figures.