ON GAS FLOWS WITH LARGE SUPERSONIC VELOCITY

1960

SovietRxiv

Abstract

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HYDROMECHANICS

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ON GAS FLOWS WITH LARGE SUPERSONIC VELOCITY

(Presented by Academician A. A. Dorodnitsyn, 27 IV 1960)

Below are written the general equations of gas flows with large supersonic velocity, which are simplified in different ways depending on the magnitude of the parameter K, analogous to the local similarity parameter of hypersonic flows ([1]). For $K \sim 1$, in the case of flow past a thin body, reduction to the equations of the theory of small disturbances of a hypersonic flow is possible ([2,3]). For $K \gg 1$ a solution of the Cauchy problem has been obtained, which may be of interest in the consideration of internal flows. In the Cauchy problem, infinite domains of definition of the solution have been found, which is a consequence of the parabolic degeneration of the equations under consideration as the Mach number tends to infinity.

§ 1. A steady flow of an ideal perfect gas with constant stagnation enthalpy, corresponding to a maximum velocity equal to unity, is considered. The velocity vector $\mathbf{v}$ is represented in the form

$$\mathbf{v} = (1-\eta)\vec{\tau}, \quad |\vec{\tau}| = 1, \quad M = \sqrt{2}\,[(\varkappa-1)\eta(2-\eta)]^{-1/2}(1-\eta), \tag{1}$$

where the relation between the Mach number M and the quantity η is given by Bernoulli' s equation, and $\varkappa$ is the adiabatic exponent. It is assumed everywhere that $M \gg 1$, i.e. $\eta \ll 1$. The expression for $\mathbf{v}$ is substituted into the equations of motion, which are transformed so that the coefficient of each derivative of the sought quantity contains a polynomial depending on η. The simplification of the equations consists in the following: in each of the polynomials we neglect the higher terms in comparison with the term containing η in the lowest degree.

As a result, in the case of plane ($\nu = 0$) and axisymmetric ($\nu = 1$) flows, the equations of continuity, momentum, and adiabaticity can be written as

$$\frac{1}{\varkappa-1}\frac{\partial \ln\eta}{\partial\tau} + \frac{\partial\theta}{\partial n} + \nu\frac{\sin\theta}{y} = 0, \qquad \frac{\partial\theta}{\partial\tau} + \frac{\partial\eta}{\partial n} - \eta\frac{\partial\sigma}{\partial n} = 0, \qquad \frac{\partial\sigma}{\partial\tau} = 0, \tag{2}$$

where σ is the entropy function (3), θ is the angle of inclination of the velocity vector to the x-axis of the Cartesian (cylindrical for $\nu = 1$, when the x-axis is the axis of symmetry) coordinate system x, y; $\partial/\partial\tau$ and $\partial/\partial n$ denote, respectively, differentiation along a streamline and along the normal to it (n is rotated relative to $\vec{\tau}$ by $\pi/2$ counterclockwise).

The pressure p and density ρ are expressed by

$$\sigma = \ln \frac{p^{1/\varkappa}}{\rho} + \text{const}, \qquad \frac{\varkappa}{\varkappa-1}\frac{p}{\rho} = \eta + O(\eta^2). \tag{3}$$

The characteristic equations of system (2) have the form (ψ is the Mach angle)

$$dy - \operatorname{tg}(\theta \pm \psi)\, dx = 0, \qquad \operatorname{tg}^2 \psi = (\varkappa - 1)\eta,$$

$$\pm d\theta + \frac{2}{\sqrt{\varkappa-1}}\, d\sqrt{\eta} + \nu \frac{\sin\theta \, \sin\psi}{\sin(\theta \pm \psi)} \frac{dy}{y} - \frac{\operatorname{tg}\psi}{\varkappa - 1}\, d\sigma = 0, \tag{4}$$

where the sign $+(-)$ refers to the characteristics of the I (II) family.

Let us note elementary exact solutions of equations (2): flow in a simple wave (for $\nu = 0,\ \sigma = \text{const}$)

$$y = x\ \operatorname{tg}(\theta \mp \psi) + Y(\theta), \qquad \theta \mp \frac{2\sqrt{\eta}}{\sqrt{\chi - 1}} = \text{const}, \tag{5}$$

where $Y(\theta)$ is an arbitrary function, and flow from a source (for $\sigma = \text{const}$)

$$\eta r^{(\chi-1)(1+\nu)} = \text{const}, \tag{6}$$

where r is the radius vector in polar (spherical) coordinates.

§ 2. Let us introduce, for estimating the terms in equations (2), characteristic quantities Δ and ϑ for η and θ, respectively. The Mach number corresponding to Δ (1) will be denoted by M_*. Let N and T be characteristic dimensions, respectively, along the streamlines and normal to them for the region of flows under consideration. Estimating the first two terms in the first equation (2), we obtain a relation between $N,\ T,\ \vartheta$:

$$\frac{1}{\chi-1}\frac{1}{\eta}\frac{\partial\eta}{\partial\tau} \sim \frac{1}{T}, \qquad \frac{\partial\theta}{\partial n} \sim \frac{\vartheta}{N}, \qquad \frac{1}{T} \sim \frac{\vartheta}{N}, \qquad N \sim T\vartheta. \tag{7}$$

Let us estimate the terms in the second equation (2), taking (7) into account:

$$\frac{\partial\theta}{\partial\tau} \sim \frac{\vartheta}{T}, \qquad \frac{\partial\eta}{\partial n} \sim \frac{\Delta}{N} \sim \frac{\Delta}{T\vartheta}, \qquad \eta\frac{\partial\sigma}{\partial n} \sim \frac{\Delta}{N} \sim \frac{\Delta}{T\vartheta}. \tag{8}$$

In estimating the entropy term it is assumed that σ changes by no more than a quantity of order unity. From (7) and (8) it follows that the ratio of the second and third terms to the first in the second equation (2) is, in order of magnitude,

$$\frac{\Delta}{T\vartheta} : \frac{\vartheta}{T} = \frac{\Delta}{\vartheta^2} \sim \frac{1}{M_*^2 \vartheta^2} = \frac{1}{K^2}. \tag{9}$$

Here K denotes a quantity analogous to the similarity parameter of hypersonic flows ([1]).

The following cases are possible:

A. $K \sim 1$. Consider the flow about a slender body (the characteristic angle of inclination of the generator to the axis $x\vartheta \ll 1$) by a hypersonic stream with the undisturbed-velocity vector directed along the x-axis. In the sense of relations (7), (8), as Δ and, correspondingly, M_*, one takes the values of Δ and M_* at some point of the disturbed flow, for example at the nose of the body behind the shock wave. In this case, as is not difficult to verify, always $K = M_* \vartheta \sim 1$.

According to (7), (8), all terms in equations (2) have the same order and therefore must be retained. One can only simplify the expressions for the operators $\partial/\partial n$ and $\partial/\partial\tau$, which take the form

$$\frac{\partial}{\partial\tau} = [1 + O(\vartheta^2)] \left(\frac{\partial}{\partial x} + \theta \frac{\partial}{\partial y} \right), \qquad \frac{\partial}{\partial n} = [1 + O(\vartheta^2)] \frac{\partial}{\partial y}. \tag{10}$$

Taking (3) and (10) into account, equations (2) may be written in the form of equations of unsteady flow, in accordance with ([2,3]). An example of an exact solution satisfying the condition $K \sim 1$ is the flow in a simple wave (5).

B. $K \gg 1$. Consider, when this condition is fulfilled, the solution of the Cauchy problem. Suppose that on a smooth arc AB there are prescribed continuous and continuously differentiable functions of the arc length s: $\theta = \theta_0(s)$, $\eta = \eta_0(s)$, $\sigma = \sigma_0(s)$. For simplicity, $\theta_0(s)$ is assumed to be a monotone function. On AB the conditions

$$|\theta_0(A) - \theta_0(B)| \gg \sqrt{\max \eta_0(s)}, \qquad \chi - \psi > 0, \tag{11}$$

are satisfied, where $\max \eta_0(s)$ is the maximum of the function $\eta_0(s)$ on AB; χ is the smallest angle between the direction of the velocity vector and the tangent to the curve AB; ψ–

* In order of magnitude these terms must be equal to one another.

as before, the Mach angle. It can be shown that the second condition (11) is a sufficient condition for $K \gg 1$ to hold throughout the entire region. In this case,

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

from the second equation (2), according to (9), to within a quantity of order K^{-2} we have $\partial\theta/\partial\tau = 0$, whence it follows that the streamlines are straight rays, along each of which, according to the third equation (2), $\sigma = \text{const}$. Let us construct the envelope of the rays (CD in Figs. 1 and 2) and denote by r the distance along a ray from the given point to the point of tangency of the ray with the envelope, and by a the distance along the ray from the point of tangency to the axis of symmetry.

Using geometrical considerations, we integrate the first equation (2), after which the solution of the Cauchy problem takes the form:

$$\eta = \eta_0 \left(\frac{r_0}{r}\right)^{\varkappa-1} \left|\frac{r_0 \pm a}{r \pm a}\right|^{\nu(\varkappa-1)}, \qquad \theta = \theta_0, \qquad \sigma = \sigma_0, \tag{12}$$

where the quantities with subscript zero, taken on AB, are constant for each ray; the sign + or − refers respectively to the cases of Fig. 1 and Fig. 2.

Fig. 1

Fig. 2

Solution (12) coincides with the solution for a hypersonic source (6) for $\nu = 0$ always, and for $\nu = 1$ in the case $r \gg a$, which confirms the validity of the asymptotic solution (4) of the problem of the outflow of a gas jet into vacuum.

Obviously, solution (12) is valid for sufficiently small η, i.e., at a sufficient distance from the envelope and from the axis of symmetry.

Cases are possible (Fig. 2) in which the quantity η changes nonmonotonically with changing r, having a maximum at $r = a/2$.

§ 3. Let us consider the question of the domain of definition of the solution. For simplicity put $\sigma = \text{const}$ and $a = 0$. From geometrical considerations, taking (12) into account,

$$\frac{\sin\theta \sin\psi}{\sin(\theta \pm \psi)} \frac{dy}{y} = \pm d\theta,$$

after which the equation of the characteristics in the plane η, θ (4) integrates to

$$\pm\theta(1+\nu) + \frac{2\sqrt{\eta}}{\sqrt{\varkappa - 1}} = \text{const}. \tag{13}$$

Fig. 3

Figure 2: Fig. 3

Taking into account $\lim_{\eta\to\infty}\eta = 0$, it follows from (11) and (13) that the characteristics AE (family II) and BF (family I) do not intersect one another, asymptotically approaching certain rays whose angles of inclination to the x-axis are equal to θ_- and θ_+ (Fig. 1). Thus solution (12) is defined in the region $ABFE$, extending to infinity. This fact appears essential for the calculation of hypersonic flow in a nozzle, the flow in which, in the region $ABFE$, can be completely determined by prescribing the initial segment $ABGH$.

(Fig. 3). The presence of infinite domains of definition of the solution, which also occurs in the general case, when $\sigma \neq$ const, $a \neq 0$, is a consequence of the parabolic degeneration of the equations as $M \to \infty$.

The character of the degeneration (substantially different from that occurring for $M = 1$) can be investigated by considering the equation for the Legendre potential Φ for a plane isentropic flow in the plane η, θ, with $\eta \ll 1$,

$$(\varkappa - 1)\eta\Phi_{\eta\eta} - \Phi_{\theta\theta} + \Phi_\eta = 0. \tag{14}$$

Using, in accordance with the theory [2,3], a solution of this equation known [5] from the theory of unsteady flows (we also note [6]), one can confirm the validity of the solution (12) for $v = 0$, $\sigma =$ const, if one sets $K \gg 1$.

Fig. 3

The appearance of infinite domains of definition of the solution in the physical plane corresponds, in the plane η, θ, to the following: the characteristics AE and BF, without intersecting each other, reach the line of parabolic degeneration $\eta = 0$.

The author expresses deep gratitude to A. A. Nikol' skii for his attention to the work and valuable comments.

Received
22 IV 1960

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