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Abstract

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GROUPS WITH ONE CLASS OF UNATTAINABLE ISOORDINAL Πd-SUBGROUPS

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§ 1. Let $\mathfrak{G}$ be a finite group, and let Π be a nonempty set of prime numbers. In what follows we shall use the notions of a Π-solvable group, a Π-separable group, and a Π-Sylow divisor of the order of a group, introduced by S. A. Chunikhin ([1–3]), as well as the following notions: a Πd-group (Πd-subgroup) is a group (subgroup) whose order is divisible by at least one prime number from Π (see ([4])); an attainable (subnormal) subgroup is a subgroup occurring in some normal series of the group ([5]); a Hall subgroup is a subgroup whose order is relatively prime to its index ([6]).

We introduce the following as well.

Definition. Subgroups of one and the same order will be called **isoordinal**. The set of all subgroups of a given group is partitioned into classes by the relation of isoordinality. The classes so obtained will be called **classes of isoordinal subgroups** (see ([11])). A class of isoordinal subgroups will be called a **class of isoordinal attainable subgroups** if all isoordinal subgroups belonging to the given class are attainable in the group.

If, among the isoordinal subgroups of a given class, there is at least one subgroup unattainable in the group, then this class will be called a **class of unattainable isoordinal subgroups**.

By $r(\mathfrak{G})$ we shall denote the number of classes of unattainable isoordinal Πd-subgroups.

§ 2. Groups all of whose Πd-subgroups are attainable were studied by us in ([7]). P. I. Trofimov ([8]), N. Ito ([9]), and E. N. Toropov ([10]) considered the question of solvability (or generalized solvability) of groups with a prescribed totality of classes of conjugate ([8,10]) and classes of isomorphic ([9]) noninvariant subgroups. The notion of classes of isoordinal subgroups introduced by us ([11]) generalizes the notions of classes of conjugate subgroups and classes of isomorphic subgroups. It should be noted that many results of the papers ([8–10]), obtained in considering classes of conjugate subgroups and classes of isomorphic subgroups, are special cases of more general theorems obtained in considering classes of isoordinal subgroups.

In ([11]) we proved the following theorems: on the solvability of a group all of whose Πd-subgroups are attainable; on Π-solvability of a group with one class of unattainable isoordinal subgroups; on Π-separability of groups with two and three classes of unattainable isoordinal Πd-subgroups. In the present note a theorem is proved on the solvability of groups with one class of unattainable isoordinal Πd-subgroups. The result obtained improves Theorem 2 of ([11]).

Since a group containing r classes of unattainable isoordinal subgroups contains no fewer than r classes of noninvariant conjugate subgroups, it follows directly from our results cited above and from the theorem below that Theorems 1-4 of ([10]) follow.

If, as Π, one takes all prime divisors of the order of the group, then we obtain theorems on the solvability of groups for which the number of classes of unattainable

isomorphic subgroups does not exceed three. From these results follows the solvability of groups for which the number of classes of noninvariant conjugate subgroups is not more than three (see ([8])).

§ 3. **Lemma 1.** *If a Hall subgroup $\mathfrak{H}$ is attainable in a group $\mathfrak{G}$, then $\mathfrak{H}$ will be a characteristic subgroup of the group $\mathfrak{G}$.*

Lemma 2. *Let, in a group $\mathfrak{G}$ of order $g = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, a Sylow subgroup $\mathfrak{P}_1$ of order $p_1^{\alpha_1}$ ($\alpha_1 > 1$) be unattainable, while a subgroup $\mathfrak{P}_1'$ of order $p_1^{\alpha_1 - 1}$ is attainable. Then in the group $\mathfrak{G}$ there exists a subgroup $\mathfrak{G}_i$, attainable in it, of order $g_i = p_1^{\alpha_1'} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ ($\alpha_i' \leqslant \alpha_i$), in which the subgroup $\mathfrak{P}_1'$ will be invariant.*

Lemma 3. *Let $\mathfrak{G}$ be a finite group of order $g = mn$, where $m > 1$ is the greatest Π-Sylow divisor of the number g. If $r(\mathfrak{G}) > 0$, then for such a group the relation $t \leqslant r(\mathfrak{G}) + 1$ holds, where t is the number of distinct prime Π-divisors of the order of the group.*

Lemma 4. *Let $\mathfrak{H}$ be a subgroup of the group $\mathfrak{G}$. Then $r(\mathfrak{H}) \leqslant r(\mathfrak{G})$.*

Lemma 5. *Let $\mathfrak{G}/\mathfrak{N}$ be the factor group of the group $\mathfrak{G}$ by a normal divisor $\mathfrak{N}$. Then $r(\mathfrak{G}/\mathfrak{N}) \leqslant r(\mathfrak{G})$.*

§ 4. **Theorem.** *A group with one class of unattainable isomorphic Πd-subgroups is solvable.*

Proof. Let $g = mn$ be the order of the group $\mathfrak{G}$, where $m > 1$ is the greatest Π-Sylow divisor of the order and $n \geqslant 1$. Let

$$n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_i^{\beta_i} \cdots q_k^{\beta_k}$$

be the canonical decomposition in the case $n > 1$. Denote by $\Omega_1, \Omega_2, \dots, \Omega_i, \dots, \Omega_k$ the Sylow subgroups of orders

$$q_1^{\beta_1}, q_2^{\beta_2}, \dots, q_i^{\beta_i}, \dots, q_k^{\beta_k},$$

respectively. As follows from Lemma 3, the order of a group with one class of unattainable isomorphic Πd-subgroups cannot be divisible by more than two distinct prime numbers from Π.

Let us consider the possible cases.

1. $m = p^\alpha$, i.e. the order g is divisible by only one prime number from Π.

1.1. The Sylow subgroup $\mathfrak{P}$ of order p^α is attainable in $\mathfrak{G}$. By Lemma 1 the subgroup $\mathfrak{P}$ will be invariant in $\mathfrak{G}$. In view of the invariance of the subgroup $\mathfrak{P}$, we can form the subgroups

$$\mathfrak{P}\Omega_1,\ \mathfrak{P}\Omega_2, \dots, \mathfrak{P}\Omega_i, \dots, \mathfrak{P}\Omega_k,$$

and among these subgroups there can be not more than one that is unattainable in the whole group $\mathfrak{G}$.

1.1.1. If all the subgroups obtained are attainable in $\mathfrak{G}$, then they will also be invariant in the whole group. Therefore the products

$$\mathfrak{P}\Omega_1\mathfrak{P}\Omega_2 = \mathfrak{P}\Omega_1\Omega_2, \dots,$$

$$\dots,\ \mathfrak{P}\Omega_1\mathfrak{P}\Omega_2 \cdots \mathfrak{P}\Omega_i = \mathfrak{P}\Omega_1\Omega_2 \cdots \Omega_i, \dots,$$

$$\mathfrak{P}\Omega_1\mathfrak{P}\Omega_2 \cdots \mathfrak{P}\Omega_i \cdots \mathfrak{P}\Omega_{k-1} = \mathfrak{P}\Omega_1\Omega_2 \cdots \Omega_i \cdots \Omega_{k-1}$$

will be normal divisors of the group $\mathfrak{G}$. In this case the group $\mathfrak{G}$ is solvable, as possessing a solvable normal series:

$$\mathfrak{G} \supset \mathfrak{P}\Omega_1\Omega_2 \cdots \Omega_i \cdots \Omega_{k-1} \supset \cdots \supset \mathfrak{P}\Omega_1\Omega_2 \cdots \Omega_i \supset \cdots \supset \mathfrak{P}\Omega_1\Omega_2 \supset \mathfrak{P}\Omega_1 \supset \mathfrak{P} \supset \mathfrak{E}.$$

1.1.2. Let there exist among the subgroups obtained one unattainable subgroup. Without loss of generality in the proof, suppose that the subgroup $\mathfrak{P}\Omega_k$ is unattainable. Then all the other subgroups of this collection are already attainable. Repeating the reasoning of case 1.1.1, we construct a solvable series of the group $\mathfrak{G}$.

1.2. The Sylow subgroup $\mathfrak{P}$ of order p^α is unattainable in $\mathfrak{G}$. Then all the other Πd-subgroups whose orders are different from p^α are attainable in $\mathfrak{G}$.

Note that $N_{\mathfrak{G}}(\mathfrak{P})$, the normalizer of the Sylow subgroup $\mathfrak{P}$ in the group $\mathfrak{G}$, coincides with $\mathfrak{P}$. Otherwise $N_{\mathfrak{G}}(\mathfrak{P})$ is an attainable subgroup in $\mathfrak{G}$, and, consequently, the Sylow subgroup $\mathfrak{P}$ itself will be an attainable subgroup, which contradicts the assumption.

We shall first show that in this case the group $\mathfrak{G}$ will satisfy the condition of p-solvability. For this, consider two possible cases:
a) $\alpha = 1$; b) $\alpha > 1$.

a) $\alpha = 1$, i.e. $g = pn$. $\mathfrak{N}_{\mathfrak{P}}^{\mathfrak{G}} = \mathfrak{P}$. Since the subgroup $\mathfrak{P}$ coincides with its normalizer and is cyclic, it is contained in the center of its normalizer. By Burnside' s theorem ([12]) the group $\mathfrak{G}$ has an invariant subgroup $\mathfrak{M}$ of index p, and the group $\mathfrak{G}$ will be p-solvable, as possessing a p-solvable series $\mathfrak{G} \supset \mathfrak{M} \supset \mathfrak{E}$.

b) $\alpha > 1$. Then $g = p^\alpha n$. Since $\alpha > 1$, $p^{\alpha-1} > 1$. The subgroup $\mathfrak{P}_1$ of order $p^{\alpha-1}$ will be attainable in $\mathfrak{G}$, since $r(\mathfrak{G}) = 1$. By Lemma 2, in $\mathfrak{G}$ there exists an attainable subgroup $\mathfrak{G}_i$ of order $g_i = p^\alpha n_1$ $(n_1 \leq n)$, in which $\mathfrak{P}_1$ will be invariant. By virtue of Lemma 4, $r(\mathfrak{G}_i) \leq 1$. If $r(\mathfrak{G}_i) = 0$, then the group $\mathfrak{G}_i$ is p-solvable by Theorem 1 ([11]). If $r(\mathfrak{G}_i) = 1$, then we form the factor group $\mathfrak{G}_i/\mathfrak{P}_1 = \mathfrak{H}$. Obviously, the order of $\mathfrak{H}$ will be

$$h = \frac{p^\alpha n_1}{p^{\alpha-1}} = pn_1.$$

By Lemma 5, $r(\mathfrak{H}) \leq 1$. Therefore the p-solvability of $\mathfrak{H}$ follows from the cases 1,1 or a) considered above, depending on whether or not $\mathfrak{H}$ contains an attainable subgroup of order p. The p-solvability of the group $\mathfrak{G}_i$ follows ([3]) from the p-solvability of $\mathfrak{P}_1$ and $\mathfrak{G}_i/\mathfrak{P}_1$. If

$$\mathfrak{G}_i \supset \mathfrak{G}_{i+1} \supset \cdots \supset \mathfrak{P}_1 \supset \mathfrak{E}$$

is a p-solvable series of the group $\mathfrak{G}_i$, then the series

$$\mathfrak{G} \supset \mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \cdots \supset \mathfrak{G}_i \supset \mathfrak{G}_{i+1} \supset \cdots \supset \mathfrak{P}_1 \supset \mathfrak{E},$$

which exists, in view of the attainability of $\mathfrak{G}_i$ in $\mathfrak{G}$, satisfies the condition of p-solvability, i.e. the group $\mathfrak{G}$ is p-solvable.

Form the set $\Pi_i = \{p, q_i\}$, where q_i is an arbitrary non-Π-divisor of the order of the group. The group $\mathfrak{G}$ will be Π_i-separable ([3]). By a theorem of S. A. Chunikhin ([2]), in the Π_i-separable group $\mathfrak{G}$ there exists a Hall Πd-subgroup $\mathfrak{B}$ of order $v = p^\alpha q_i^{\beta_i}$. Since the order v is different from p^α, the Hall Πd-subgroup $\mathfrak{B}$ will be attainable and, consequently, invariant in $\mathfrak{G}$. As we showed earlier,

$$\mathfrak{N}_{\mathfrak{P}}^{\mathfrak{G}} = \mathfrak{P},$$

therefore $\mathfrak{N}_{\mathfrak{P}}^{\mathfrak{G}}$ is contained in $\mathfrak{B}$.

Thus we have arrived at the conclusion that the subgroup $\mathfrak{B}$, containing the normalizer of a non-invariant Sylow subgroup $\mathfrak{P}$, is invariant in $\mathfrak{G}$. But this, for $\mathfrak{B} \neq \mathfrak{G}$, contradicts a known proposition ([13]) of O. Yu. Schmidt. Consequently, the subgroup $\mathfrak{B}$ coincides with $\mathfrak{G}$, and the order of the group $\mathfrak{G}$ has the form $g = p^\alpha q^\beta$. In this case the group $\mathfrak{G}$ is solvable as a group whose order is divisible by only two distinct prime numbers.

2. $m = p_1^{\alpha_1} p_2^{\alpha_2}$, i.e. g is divisible by two distinct prime numbers from Π. In view of the condition of the theorem, a Sylow subgroup of one of the orders $p_1^{\alpha_1}$ or $p_2^{\alpha_2}$ is attainable in $\mathfrak{G}$. Let the subgroup $\mathfrak{P}_1$ of order $p_1^{\alpha_1}$ be attainable. $\mathfrak{P}_1$ will also be invariant in $\mathfrak{G}$ (Lemma 1). The factor group $\mathfrak{G}/\mathfrak{P}_1 = \mathfrak{H}$, by Lemma 5, has no more than one class of unattainable isoordinal Πd-subgroups. If $r(\mathfrak{H}) = 0$, then the solvability of $\mathfrak{H}$ follows from Theorem 1 ([11]). If $r(\mathfrak{H}) = 1$, then the solvability of $\mathfrak{H}$ was proved above. In both cases the group $\mathfrak{G}$ will be solvable.

The theorem is proved completely.

As follows from the proof, the theorem as formulated is also true in the case when all prime divisors of the order of the group belong to the set Π. Therefore from the theorem there follows

Corollary. A group with one class of unattainable isoordinal subgroups is solvable.

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Note: Figure translations are in progress. See original paper for figures.