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HYDROMECHANICS

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

HYDROMECHANICS

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ON THE QUESTION OF THE STABILITY OF A VISCOUS FILM ON A SOLID BODY IN A GAS FLOW

(Presented by Academician I. I. Artobolevskii, 5 XI 1959)

§ 1. **Statement of the problem.** On the plane $y = 0$ (Fig. 1) there flows a layer $0 \leqslant y \leqslant h$ of a viscous heavy liquid with a linear velocity distribution

$$U = \frac{U_0}{h} y, \qquad V = 0. \tag{1}$$

In the half-space $y > h$ there flows a viscous heavy gas with velocity

$$U_1 = U_1(y), \qquad V_1 = 0. \tag{2}$$

Fig. 1

All parameters of the flow under study do not depend on the coordinate x. We shall denote parameters pertaining to the gas by the subscript 1, leaving without a subscript the parameters pertaining to the liquid layer. The pressure in the liquid and in the gas is distributed as follows:

$$P = P_0 - \rho g(y - h), \qquad P_1 = P_0 - \int_0^y \rho_1(y) g \, dy, \tag{3}$$

where ρ and ρ_1 are the densities of the liquid and gas; g is the acceleration of gravity. On the interface $y = h$ the conditions

$$U_1(h) = U_0, \qquad \mu_1 \frac{dU_1}{dy} = \mu \frac{U_0}{h}, \tag{4}$$

are satisfied, where μ and μ_1 are the corresponding viscosity coefficients.

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We superpose small disturbances on the basic flow

$$u = U + u', \quad u_1 = U_1 + u_1', \quad v = v', \quad v_1 = v_1', \quad p = P + p', \quad p_1 = P_1 + p_1', \tag{5}$$

whereupon the former interface $y = h$ assumes the form

$$y = h + h'(x, t). \tag{6}$$

On the interface we require the relations

$$u' = u_1', \quad v' = v_1', \quad v' = \frac{\partial h'}{\partial t} + U_0 \frac{\partial h'}{\partial x}, \quad p_\tau = p_{\tau 1}, \quad p - p_{n1} = K \frac{\partial^2 h'}{\partial x^2}, \tag{7}$$

where p_τ are the tangential stresses, and p_n the normal stresses on the interface; K is the coefficient of surface tension.

Introducing dimensionless parameters and the stream function ψ,

$$\begin{gathered} \varepsilon = \mu_1/\mu, \quad \sigma = \rho_1/\rho, \quad T = K\rho h/\mu^2, \\ G = \rho^2 g h^3/\mu^2, \quad \mathrm{Re} = U_0 h \rho/\mu, \quad N = \mu\rho_1/\mu_1\rho = \sigma/\varepsilon \end{gathered} \tag{8}$$

$$u' = \partial\psi/\partial y, \quad v' = -\partial\psi/\partial x, \tag{9}$$

we consider a particular form of the disturbances

$$h' = \delta e^{i\alpha(x-ct)}, \quad \psi = \delta f(y) e^{i\alpha(x-ct)}, \quad u_1' = \delta u_1(y) e^{i\alpha(x-ct)}, \tag{10}$$

$$v' = \delta U_1(y) e^{i\alpha(x-ct)}, \quad p' = \delta p(y) e^{i\alpha(x-ct)}, \quad p_1' = \delta p(y) e^{i\alpha(x-ct)},$$

where x and y are referred to h; t to h/U_0; u' and v' to U_0; p to $\mu U_0/h$.

Using the linearized Navier–Stokes equations and all the preceding assumptions, we reduce the boundary conditions (7) to the form

$$f'(1) = u_1(1), \quad -i\alpha f(1) = v_1(1), \tag{11}$$

$$f(1) = c - 1, \tag{12}$$

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$$f''(1) + \alpha^2(c-1) = \varepsilon\,[u_1'(1) + \alpha^2(c-1)],$$

$$f'''(1) - 3\alpha^2 f'(1) + i\alpha\,\mathrm{Re}(c-1)[f'(1) + 1 - N] + \varepsilon[2\alpha^2 f'(1) -$$

$$-u_1''(1) + i\alpha v_1'(1)] - \sigma i\alpha\,\mathrm{Re}(c-1) f'(1) = i\alpha[G(1-\sigma) + \alpha^2 T]\frac{1}{\mathrm{Re}}. \qquad (13)$$

If now ε and σ are made to tend to zero in such a way that the limit $\lim\limits_{\varepsilon\to 0,\ \sigma\to 0}(\sigma/\varepsilon) = N$ remains some finite quantity, then the boundary conditions (13) will not contain parameters characterizing the motion of the gas:

$$f''(1) = -\alpha^2(c-1),$$

$$f'''(1) - 3\alpha^2 f'(1) + i\alpha\,\mathrm{Re}(c-1)[f'(1) + 1 - N] = i\alpha(G + \alpha^2 T)\frac{1}{\mathrm{Re}}. \qquad (14)$$

The latter expressions may be regarded as the result of expanding the quantities entering into conditions (13) in powers of the small parameters ε and σ, retaining only the first terms of the expansion.

At the solid wall we take the no-slip condition for the liquid

$$f(0) = f'(0) = 0. \qquad (15)$$

Thus, the problem of the stability of a thin liquid film on a body in a gas stream, for sufficiently small ε and σ, reduces to the study of a particular form of the Orr–Sommerfeld equation ([1])

$$f^{\mathrm{IV}}(y) - 2\alpha^2 f''(y) + \alpha^4 f(y) - i\alpha\,\mathrm{Re}(y-c)[f''(y) - \alpha^2 f(y)] = 0 \qquad (16)$$

with boundary conditions (12), (14), (15).

§ 2. **Characteristic equation.** We shall seek the solution of equation (16) in the form of a series in powers of the parameter $\lambda = \alpha\,\mathrm{Re}$

$$f(y) = \sum_{n=0}^{\infty} \lambda^n f_n(y). \qquad (17)$$

Substituting (17) into equation (16) and equating coefficients of equal powers of λ, we obtain the system of differential equations

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Fig. 2

Figure 2: Fig. 2

$$f_n^{\mathrm{IV}} - 2\alpha^2 f_n'' + \alpha^4 f_n = F_n, \quad n = 0, 1, 2, \dots, \tag{18}$$

where

$$F_0 = 0, \quad F_n = i(y - c)[f_{n-1}'' - \alpha^2 f_{n-1}], \quad n = 1, 2, 3, \dots \tag{19}$$

This system is easily integrated in elementary functions successively, starting with $n = 0$. In order that the function $f(y)$ satisfy the boundary conditions (12), (14), (15), it is sufficient to require the fulfillment of the following-

following relations for the solution of the system of equations (18), (19):

$$f_n(0) = f_n'(0) = 0, \quad n = 0, 1, 2, \dots, \quad f_0(1) = c - 1, \quad f_0''(1) = -\alpha^2(c-1);$$
$$f_n(1) = f_n'(1) = 0, \quad n = 1, 2, 3, \dots; \tag{20}$$

$$f_0'''(1) - 3\alpha^2 f_0'(1) - i\alpha\,(G + \alpha^2 T)\frac{1}{\mathrm{Re}} + i\lambda(c-1)(1-N) +$$
$$+ \sum_{n=1}^{\infty} \lambda^n \left[f_n'''(1) - 3\alpha^2 f_n'(1) + i(c-1) f_{n-1}'(1)\right] = 0. \tag{21}$$

The general solution of the system of differential equations (18), (19) is

$$f_n(y) = \Phi_{n1}(y)\,\mathrm{sh}\,\alpha y + \Phi_{n2}(y)\,\mathrm{ch}\,\alpha y + \Phi_{n3}(y) y\,\mathrm{sh}\,\alpha y + \Phi_{n4}(y) y\,\mathrm{ch}\,\alpha y +$$
$$+ C_{n1}\,\mathrm{sh}\,\alpha y + C_{n2}\,\mathrm{ch}\,\alpha y + C_{n3} y\,\mathrm{sh}\,\alpha y + C_{n4} y\,\mathrm{ch}\,\alpha y, \tag{22}$$

where

$$\Phi_{n1} = -\frac{1}{2\alpha^3}\int_0^y F_n(\eta)(\mathrm{ch}\,\alpha\eta - \alpha\eta\,\mathrm{sh}\,\alpha\eta)\,d\eta,$$
$$\Phi_{n2} = \frac{1}{2\alpha^3}\int_0^y F_n(\eta)(\mathrm{sh}\,\alpha\eta - \alpha\eta\,\mathrm{ch}\,\alpha\eta)\,d\eta, \tag{23}$$
$$\Phi_{n3} = -\frac{1}{2\alpha^2}\int_0^y F_n(\eta)\,\mathrm{sh}\,\alpha\eta\,d\eta, \quad \Phi_{n4} = \frac{1}{2\alpha^2}\int_0^y F_n(\eta)\,\mathrm{ch}\,\alpha\eta\,d\eta.$$

Fig. 2

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C_{nk} are constants of integration; $F_n(\eta)$ is determined by formulas (19).

Using the boundary conditions (20), we find the constants of integration C_{nk}, $n = 0, 1, 2, \ldots;\ k = 1, 2, 3, 4$. Substituting the solution thus obtained, which contains no constants of integration, into relation (21), we find the characteristic equation. Neglecting in this equation the terms containing powers of λ higher than the second, we obtain the approximate characteristic equation

$$(c-1)\left[Q_{00} + \lambda^2(Q_{22}c^2 + Q_{21}c + Q_{20})\right] + \\ +i\left[\lambda(c-1)(Q_{11}c + Q_{10} + 1 - N) - \frac{\alpha}{\mathrm{Re}}(G + \alpha^2 T)\right] = 0, \tag{24}$$

where Q_{ik} are real quantities depending only on α.

§ 3. **Neutral curves, critical Reynolds number.** Taking c to be real, we separate in equation (24) the real and imaginary parts; then we obtain the system of equations

$$(c-1)\left[Q_{00} + \alpha^2\,\mathrm{Re}^2(Q_{22}c^2 + Q_{21}c + Q_{20})\right] = 0, \tag{25}$$

$$(c-1)(Q_{11}c + Q_{10} + 1 - N)\,\mathrm{Re}^2 - (G + \alpha^2 T) = 0. \tag{26}$$

If $G \equiv T \equiv 0$, then (25) and (26) are satisfied for $c = 1$ for arbitrary Re and α, but in this case the imaginary part of c is identically equal to zero, whereas loss of stability occurs when the sign of the imaginary part of c changes. Therefore we shall exclude this case from consideration.

If $c \neq 1$, then from equation (25) we find

$$\mathrm{Re} = \sqrt{-\frac{Q_{00}}{\alpha^2\left(Q_{22}c^2 + Q_{21}c + Q_{20}\right)}}. \tag{27}$$

Substituting (27) into (26), we obtain a quadratic equation with respect to c

$$c^2\left[Q_{00}Q_{11} + \alpha^2 Q_{22}\left(G + \alpha^2 T\right)\right] + c\left[Q_{00}\left(Q_{10} + 1 - N - Q_{11}\right) + \alpha^2 Q_{21}\left(G + \alpha^2 T\right)\right] \\ - Q_{00}\left(Q_{10} + 1 - N\right) - \alpha^2 Q_{20}\left(G + \alpha^2 T\right) = 0. \tag{28}$$

The roots of this equation are c_1 and c_2. Substituting c_1 and c_2 into (27), we find, respectively, Re_1 and Re_2. The curves of the dependence of Re on α are customarily called

Fig. 3

neutral. In the case $G \geq 0$, the neutral curves have the form shown in Fig. 2 ($N = 0,\ T = 500$). The minimum value $\min \mathrm{Re}(\alpha) = \mathrm{Re}_{\mathrm{cr}}$ is called the critical

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Fig. 3

Figure 3: Fig. 3

Reynolds number. In the case $G < 0$ we shall not have $\mathrm{Re}_{\mathrm{cr}}$ without some additional condition, since in this case, in addition to the neutral curves of the former form, there is the straight line

$$\alpha = \sqrt{-G/T},$$

and the region of instability will lie below this straight line. In the subsequent calculations we set $G = 0$. This is justified by the fact that G enters all formulas only in the form of the combination $G + \alpha^2 T$; thus, the case $G > 0$ can be obtained from the case $G = 0$ by a corresponding increase in T.

Figure 3 gives the results of calculating $\mathrm{Re}_{\mathrm{cr}}$ as a function of the parameters T and N (a— $\mathrm{Re}_{1\mathrm{cr}}$, b— $\mathrm{Re}_{2\mathrm{cr}}$, c— $\mathrm{Re}_{\mathrm{cr}}$).

The Reynolds number can be written in the form $\mathrm{Re} = \rho h^2 \tau / \mu^2$, where $\tau = \mu U_0 / h$ is the shear stress on the surface of the layer. The value of τ can be calculated using boundary-layer theory in the gas, or found from experiment; then, knowing the critical Reynolds number, we can determine the thickness of the stable liquid layer on the body.

The calculations were carried out on the "Strela" electronic computer.

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REFERENCES

1. **Lin Chia-chiao**, *Theory of Hydrodynamic Stability*, IL, 1958.

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