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Fig. 1

Figure 1: Fig. 1

**Abstract**

**Full Text**

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**THEORY OF ELASTICITY**

**D. D. IVLEV**

**ON THE EQUATIONS OF LINEARIZED  
SPATIAL PROBLEMS IN THE THEORY OF  
IDEAL PLASTICITY**

*(Presented by Academician Yu. N. Rabotnov on 2 VII 1959)*

Planar linearized problems in the theory of an ideally rigid-plastic body were considered in the works (<sup>1</sup>, <sup>2</sup>). In both cases the initial relations led to a wave equation. In the present note the equations of the simplest linearized spatial problems are considered.

Imagine a bar of square cross-section, weakened by smooth notches. Direct the  $z$ -axis along the axis of the bar, and the  $x$ - and  $y$ -axes perpendicular to its faces (Fig. 1). Consider the plastic flow of the bar under the action of tensile forces acting along the  $z$ -axis. Denote by  $2a$  the length of the sides of the square lying at the base of the bar. We shall prescribe the equations of the sides of the bar in the form

**Fig. 1**

$$x = \pm[a - \delta f(z)], \quad y = \pm[a - \delta f(z)], \quad (1)$$

where  $\delta$  is a small parameter characterizing the depth of the notch.

The solution should be sought in the form

$$\sigma_x = \sigma_x^0 + \delta \sigma'_x, \dots; \quad \varepsilon_x = \varepsilon_x^0 + \delta \varepsilon'_x, \dots; \quad u = u^0 + \delta u', \dots, \quad (2)$$

where  $\sigma_x, \dots$  are stress components;  $\varepsilon_x, \dots$  are strain components;  $u, \dots$  are components of the displacement velocity.

Assuming that the material of the bar is ideally plastic, we write the relations generalizing M. Lévy' s substitution in the form <sup>(3)</sup>

$$\begin{aligned} \sigma_x &= \sigma - \frac{2}{3}k + 2k \cos^2 \varphi_1, & \sigma_y &= \sigma - \frac{2}{3}k + 2k \cos^2 \varphi_2, \\ \sigma_z &= \sigma - \frac{2}{3}k + 2k \cos^2 \varphi_3, & \tau_{xy} &= 2k \cos \varphi_1 \cos \varphi_2, \\ \tau_{yz} &= 2k \cos \varphi_2 \cos \varphi_3, & \tau_{zx} &= 2k \cos \varphi_3 \cos \varphi_1, \end{aligned} \quad (3)$$

where  $k$  is the yield limit in shear,  $\sigma = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$ .

In the case under consideration the solution should be sought near the unperturbed state

$$\sigma_z^0 = 2k, \quad \sigma_x^0 = \sigma_y^0 = \tau_{xy}^0 = \tau_{yz}^0 = \tau_{zx}^0 = 0; \quad (4)$$

$$\varepsilon_x^0 = \varepsilon_y^0 = \varepsilon_z^0 = \varepsilon_{xy}^0 = \varepsilon_{yx}^0 = \varepsilon_{zx}^0 = u^0 = v^0 = w^0 = 0. \quad (5)$$

It is obvious that  $\varphi_1^0 = \varphi_2^0 = \frac{\pi}{2}$ ,  $\varphi_3^0 = 0$ . Substituting the expansions of the compo-

...the stress components in expressions (3), we obtain

$$\sigma'_x = \sigma'_y = \sigma'_z = \sigma', \quad \tau'_{xy} = 0. \quad (6)$$

The equilibrium equations take the form

$$\frac{\partial \sigma'}{\partial x} + \frac{\partial \tau'_{xz}}{\partial z} = 0, \quad \frac{\partial \sigma'}{\partial y} + \frac{\partial \tau'_{yz}}{\partial z} = 0, \quad \frac{\partial \tau'_{xz}}{\partial x} + \frac{\partial \tau'_{yz}}{\partial y} + \frac{\partial \sigma'}{\partial z} = 0. \quad (7)$$

Putting

$$\sigma' = \frac{\partial U}{\partial z}, \quad \tau'_{xz} = -\frac{\partial U}{\partial x}, \quad \tau'_{yz} = -\frac{\partial U}{\partial y},$$

from (7) we obtain

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 U}{\partial z^2} = 0. \quad (8)$$

Let us next consider the boundary conditions. Since the lateral faces are free of stresses, we have

Fig. 2

Figure 2: Fig. 2

$$\begin{aligned} \sigma_x \sin(nz) + \tau_{xz} \cos(nz) &= 0, \\ \tau_{xz} \sin(nz) + \sigma_z \cos(nz) &= 0, \end{aligned} \quad \text{for } x = \pm[\alpha - \delta f(z)], \quad (9)$$

where the angle  $(nz)$  is formed by the direction of the normal to the surface of the face and the  $z$ -axis. Analogous relations hold on the other two faces.

Linearizing the boundary conditions, we obtain

$$\begin{aligned} \sigma'_x = 0, \quad \tau'_{xz} \pm 2k \frac{df}{dz} = 0, \quad x = \pm a, \\ \sigma'_y = 0, \quad \tau'_{yz} \pm 2k \frac{df}{dz} = 0, \quad y = \pm a. \end{aligned} \quad \text{for} \quad (10)$$

We rewrite expressions (10) in the form

$$\begin{aligned} \frac{\partial U}{\partial z} = 0, \quad \frac{\partial U}{\partial x} = \mp 2k \frac{df}{dz}, \quad x = \pm a, \\ \frac{\partial U}{\partial z} = 0, \quad \frac{\partial U}{\partial y} = \mp 2k \frac{df}{dz}, \quad y = \pm a. \end{aligned} \quad \text{for} \quad (11)$$

Let us determine the domain in which the solution is to be defined. Suppose that  $f(z) = f(-z)$ , and that  $f(z)$  is a monotone function whose derivatives are sufficiently small. Figure 2 shows the section of the bar by the plane  $z = 0$ . The domain of definition is symmetric with respect to this plane. By virtue of symmetry with respect to the planes  $x = 0$  and  $y = 0$ , the solution should be sought in the regions  $ABFEO$ ,  $BCKFO$ ,  $CDLKO$ ,  $ADLEO$ . It is evident that  $AE = BF = CK = DL$ , and the planes  $AEO$ ,  $BFO$ ,  $CKO$ ,  $DLO$  form with the  $z$ -axis an angle equal to  $\pi/4$ .

Fig. 2

An essential feature of the solution is the necessity of satisfying the conditions for conjugation of solutions. Indeed, on the planes  $AEO$ ,  $BFO$ ,  $CKO$ ,  $DLO$  the conjugation conditions for the normal and tangential stress components must be fulfilled. The normal stress components are conjugated automatically, while the conjugation conditions for the tangential components lead to the necessity that the shear stresses acting on these facets vanish:

$$\tau'_{xz} - \tau'_{yz} = 0 \quad \text{for } x \pm y = 0,$$

whence

$$\frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = 0, \quad \text{for } x \pm y = 0. \quad (12)$$

Let us next consider the equations determining the displacement-velocity field. We write the equations determining plastic flow in the form <sup>(4)</sup>

$$\begin{aligned} \varepsilon_x + \frac{\cos \varphi_2}{\cos \varphi_1} \varepsilon_{xy} + \frac{\cos \varphi_3}{\cos \varphi_1} \varepsilon_{xz} = \\ = \varepsilon_{xy} \frac{\cos \varphi_1}{\cos \varphi_2} + \varepsilon_y + \varepsilon_{yz} \frac{\cos \varphi_3}{\cos \varphi_2} = \varepsilon_{xz} \frac{\cos \varphi_1}{\cos \varphi_3} + \varepsilon_{yz} \frac{\cos \varphi_2}{\cos \varphi_3} + \varepsilon_z; \end{aligned} \quad (13)$$

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0. \quad (14)$$

Linearizing conditions (13), taking into account expressions (5), we obtain  $\varepsilon'_{xz} = \varepsilon'_{yz} = 0$ , or

$$\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} = 0, \quad \frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y} = 0. \quad (15)$$

Putting

$$u' = \frac{\partial W}{\partial x}, \quad v' = \frac{\partial W}{\partial y}, \quad w' = -\frac{\partial W}{\partial z},$$

from the incompressibility equation (14) we obtain

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial z^2} = 0. \quad (16)$$

Let us consider the boundary conditions. Suppose that the bar is stretched with unit velocity  $w' = 1$ . The normal component of the velocity to the boundary separating the regions of plastic and rigid states of the material must be equal to zero; therefore

$$\begin{aligned} \frac{\partial W}{\partial x} + \frac{\partial W}{\partial z} = 0 \\ \frac{\partial W}{\partial y} + \frac{\partial W}{\partial z} = 0 \end{aligned} \quad \text{for} \quad \begin{aligned} x \pm z = 0, \\ y \pm z = 0 \end{aligned}$$

in the corresponding domains of definition. The construction of the velocity field can be carried out following the ideas of <sup>(1)</sup>, taking in the corresponding regions the quantity  $W$  to depend only on two variables.

Fig. 3

Figure 3: Fig. 3

Fig. 3

Let us next consider the problem of indenting a thin, well-lubricated blade into a plastic half-space (Fig. 3). We write the equation of the blade in the form

$$z = \delta F(x, y, t), \quad (17)$$

where  $t$  is time, entering as a parameter. We shall assume the function  $F$  to be symmetric with respect to the  $y$ -axis. Obviously, the stressed and deformed state should be sought in the form (2), (4), (5); therefore the problem is again reduced to determining solutions of the wave equations (8), (16).

During indentation the plastic material will bulge; the surface of the bulged material  $\Phi(z, y)$  is unknown and is to be determined. On the boundary of the blade the tangential stresses are zero; on the surface  $\Phi$  the normal and tangential stresses are zero. Since the boundary conditions for the stresses contain the equation of the unknown surface  $\Phi$ , the solution of the problem in this case must begin with determining

of the kinematic picture of the phenomenon. The boundary conditions for the displacement velocities are obtained from the following conditions. During indentation the point  $M(x, y, 0)$  passes to the point  $M'(x + u', y + v', w')$ . The point  $M'$  lies on the surface of the blade; therefore  $w' \simeq F(x, y, t)$ . The desired boundary condition is written in the form

$$\frac{\partial W}{\partial z} = -\dot{\delta} F(x, y, t) \quad \text{for } z = 0.$$

Next it must be taken into account that, at the boundary between the rigid and plastic states of the material, the normal component of the displacement velocity must be equal to zero.

Only after determining the surface  $\Phi$  of the bulged material can the stress field be determined.

The problems posed are characterized by the fact that the boundary conditions for the wave equations are formulated not on one but on several planes or surfaces.

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## REFERENCES

- <sup>1</sup> E. Onat, W. Prager, *J. Appl. Phys.*, **25**, No. 4 (1954); *Collection of Translations, Mechanics*, No. 4, 1955.
- <sup>2</sup> D. D. Ivlev, *Izv. AN SSSR, OTN*, No. 10 (1957).
- <sup>3</sup> D. D. Ivlev, *Prikl. matem. i mekh.*, **22**, issue 1 (1958).
- <sup>4</sup> D. D. Ivlev, *DAN*, **124**, No. 3 (1959).

*Note: Figure translations are in progress. See original paper for figures.*

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