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MATHEMATICS

I. A. SOKOLENKO

SPHERES AND GEODESICS IN RIEMANNIAN SPACES WITH A POLE

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We consider a Riemannian space of positive curvature R^m with a pole O (^{1}). The locus of points of R^m at distance ρ from O will be called the **sphere of radius ρ with center at O** and denoted by S_ρ. $R^m \setminus S_\rho$ consists of two components; the bounded component will be called the **ball with center O of radius ρ** and denoted by Π_ρ, and the unbounded component will be denoted by T_ρ. The sphere S_ρ is called **strictly convex** if, for any points $M_1, M_2 \in S_\rho$, every shortest path M_1M_2 lies in Π_ρ, except for its endpoints.

For spaces R^m satisfying the above conditions, the following theorems are proved.

Theorem 1. *All spheres of R^m with center at the pole O are strictly convex.*

We note that from the proof given below there follows a stronger assertion: every geodesic arc (not necessarily a shortest one) with endpoints on S_ρ lies in Π_ρ, except for its endpoints.

The following propositions on geodesics in R^m follow from Theorem 1.

Theorem 2. *In the space R^m there are no closed geodesics.*

Theorem 3. *If the curvature of R^m at every point and in every two-dimensional direction is not greater than H, then every geodesic arc of length $\pi/\sqrt{H}$ is a shortest path.*

The assertion of Theorem 3 was proved by A. V. Pogorelov (^{2}) for closed two-dimensional convex surfaces and was extended by Klingenberg (^{3}) to compact spaces of positive curvature of even dimension.

Proof of Theorem 1. Let $M_1, M_2 \in S_\rho$, but not all other points of the shortest path M_1M_2 belong to Π_ρ. Then the following cases are possible:

I. M_1M_2 contains some arc $N_1N_2 \subset \overline{T}_\rho$ ($\overline{T}_\rho$ is the closure of T_ρ). Joining O to N_1 and N_2 by shortest paths, we obtain the triangle ON_1N_2. The angles

ON_1N_2 and ON_2N_1, obviously, are not less than $\pi/2$, which contradicts the comparison theorem (^{1}).

II. The shortest path $M_1M_2 \subset \overline{\Pi}_\rho$, and $M_1M_2 \cap S_\rho$ is a closed nowhere dense set on M_1M_2, not reducible to the two points M_1, M_2. Then there exist at least two distinct intervals complementary to this set. Choose on these intervals one point each, so that these points N_1, N_2 lie on some sphere S_{ρ_1}, $\rho_1 < \rho$; the existence of such points follows easily from the continuity of distance. The arc $N_1N_2 \subset M_1M_2$ is a shortest path and, according to what was proved in case I, cannot contain points of T_{ρ_1}; consequently, $N_1N_2 \subset \overline{\Pi}_{\rho_1} \subset \Pi_\rho$. This contradicts the definition of the points N_1, N_2, and Theorem 1 is proved.

Proof of Theorem 2. Suppose that in R^m there exists a closed geodesic. Let $\rho = \max \rho(O, P)$, where $P \in g$, and $\rho(O, P)$ is the dist–

distance in R^m. Then $g \subset \overline{\Pi}_\rho$ and $g \cap S_\rho \neq 0$. By Theorem 1, $g \cap S_\rho$ cannot contain any arc and, consequently, is a closed nowhere dense subset of g. In the complementary set $g \setminus (g \cap S_\rho)$ one can choose points N_1, N_2 with the same properties as in case II of the proof of Theorem 1, and so close that the smaller arc $N_1N_2 \subset g$ is shortest. This leads to a contradiction, just as in the proof of Theorem 1.

Proof of Theorem 3. Suppose that the assertion of Theorem 3 is false; then there exists a geodesic arc g_1 of length $\pi/\sqrt{H}$ which is not shortest. With the aid of Jacobi' s equations it is proved that no geodesic arc of length less than $\pi/\sqrt{H}$ contains a pair of conjugate points.

Let g_2 be a shortest arc joining the endpoints of g_1; then g_1, g_2 form a geodesic digon of perimeter less than $2\pi/\sqrt{H}$. Take a closed ball $\overline{\Pi}_\rho$ containing the constructed digon. Denote by Ω the set of all geodesic digons of perimeter less than $2\pi/\sqrt{H}$ lying in $\overline{\Pi}_\rho$; evidently, Ω is nonempty. The subsequent arguments are built on combining certain methods of A. V. Pogorelov (^{2}) with our Theorems 1 and 2.

Let s_0 be the lower bound of the lengths of the digons in Ω; let h_n be a sequence of digons in Ω whose lengths converge to s_0. One can choose h_n in such a way that their constituent geodesic arcs all have length less than $\pi/\sqrt{H}$ and converge to some geodesic arcs g_3, g_4. Since arcs of length less than $\pi/\sqrt{H}$ contain no conjugate points, g_3 and g_4 do not coincide and form a certain digon $g \in \Omega$. We shall show that g is a closed geodesic; then we arrive at a contradiction with Theorem 2, and Theorem 3 will be proved. If g is not a closed geodesic, then at one of the vertices of the digon g, for example at P_1, the arcs g_3, g_4 form an angle smaller than π. Denote by P_2 the second vertex of the digon g. By virtue of the absence on g_3 and g_4 of points conjugate to P_2, one can surround g_3 and g_4 with central fields of geodesics G_3 and G_4 in such a way that some neighborhood U of the point P_1 belongs to both fields. Take on g_3, g_4, respectively, points P_3, P_4, distinct from P_1 and so close to P_1 that the shortest arc P_3P_4 belongs to U. Then, by Theorem 1, the midpoint P_5 of the shortest arc P_3P_4 belongs to Π_ρ. Since, moreover, $P_5 \in U$, in the fields G_3, G_4 there are contained, respectively, geodesic arcs g_3', g_4', joining P_2 with P_5; by Theorem 1, g_3', g_4' lie in Π_ρ. By Weierstrass' s theorem,

$$\begin{aligned}&\text{the length of } g_3' \text{ is less than the length of } P_2P_3 + \text{the length of } P_3P_5,\\ &\text{the length of } g_4' \text{ is less than the length of } P_2P_4 + \text{the length of } P_4P_5.\end{aligned} \tag{1}$$

Since the angle at the point P_1 is less than π, we have

$$\text{the length of } P_3P_4 \text{ is less than the length of } P_3P_1 + \text{the length of } P_4P_1. \tag{2}$$

From (1), (2) it follows that the perimeter of the geodesic digon $g' = (g_3', g_4')$ is less than s_0; but g' belongs to Ω, which contradicts the definition of s_0. The contradiction obtained proves Theorem 3.

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Novosibirsk State
Pedagogical Institute

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Note: Figure translations are in progress. See original paper for figures.

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