Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1960. Volume 132, No. 4

MATHEMATICS

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A CONGRUENCE OF CENTROIDAL GEODESIC CURVES OF A METRIC SPACE OF LINEAR ELEMENTS

(Presented by Academician P. S. Aleksandrov, 11 II 1960)

1. In the present work the differential geometry is constructed of congruences of centroidal geodesics of a metric space of linear elements with Euclidean connection, whose base space is three-dimensional ($\mathfrak{F}_3$). Principal attention is devoted to those concepts that generalize the basic concepts of the classical theory of congruences of straight lines in Euclidean space $\mathbf{E}_3$ ([1]). The constructions are carried out by the group-theoretic method of G. F. Laptev ([2]) and have an invariant character. Since a Finsler space $\mathbf{F}_3$ and a space of Euclidean connection $\mathbf{V}_3$ (a Riemannian space with torsion) are special cases of the spaces $\mathfrak{F}_3$, the constructions apply both to congruences of centroidal geodesic curves of the space $\mathbf{F}_3$ and to congruences of geodesic spaces of $\mathbf{V}_3$.

2. Let $\mathbf{L}_3$ be a space of linear elements (u, v), whose base space is three-dimensional. To each linear element (u, v) of the space $\mathbf{L}_3$ we attach an affine frame $\{A(u), e_i(u, v)\}$, whose vertex coincides with the center of the linear element. Thus, with each linear element (u, v) there is associated a centro-affine space $\mathbf{A}_3(u, v)$ (a local space or layer). Suppose that the vector e_1 has the direction of the linear element, i.e. that $v^1 = 1$, $v^2 = v^3 = 0$ for the initial local space.

We shall assume that an affine connection is introduced in this manifold of centro-affine spaces $\{\mathbf{A}_3\}$. Then the mapping, defining the connection, of the neighboring local space $\mathbf{A}_3 + d\mathbf{A}_3$ onto the initial local space $\mathbf{A}_3$ will be determined by the mapping of the initial local frame $\{A(t + dt), e_i(t + dt)\}^*$ of the space $\mathbf{A}_3 + d\mathbf{A}_3$ into some frame $\{A(t, dt), e_i(t, dt)\}$ of the space $\mathbf{A}_3$

$$A(t + dt) \to A(t, dt) \simeq \omega^i e_i(t),$$

$$e_i(t + dt) \to e_i(t, dt) \simeq e_i(t) + \omega_i^k e_k(t) \tag{1}$$

$$(i, j, k = 1, 2, 3; \quad \alpha, \beta, \gamma = 2, 3; \quad p, q = 1, 2, \dots, 5).$$

The structure equations of a space of linear elements with affine connection have the form

$$D\omega^i = [\omega^k, \omega_k^i] + \Omega^i, \qquad D\omega_j^i = [\omega_j^k, \omega_k^i] + \Omega_j^i, \tag{2}$$

where

$$\Omega^i = R_{jk}^i[\omega^j, \omega^k] + C_{j\alpha}^i[\omega^j, \omega_1^\alpha],$$

$$\Omega_j^i = R_{jkl}^i[\omega^k, \omega^l] + S_{jk\alpha}^i[\omega^k, \omega_1^\alpha] + P_{j\alpha\beta}^i[\omega_1^\alpha, \omega_1^\beta],$$

$$R_{(jk)}^i = 0, \qquad R_{j(kl)}^i = 0, \qquad P_{j(\alpha\beta)}^i = 0.$$

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* t^p denote nonhomogeneous coordinates of the linear element.

and the Pfaffian forms ω^i and ω_1^α form a basis of linear differential forms of the space of linear elements. The coordinates u^i of the center of a linear element are the first integrals of the system $\omega^i = 0$, and the nonhomogeneous coordinates t^ρ of the linear element are the first integrals of the system $\omega^i = \omega_1^\alpha = 0$, with $t^i = u^i$.

3. With the aid of a symmetric covariant tensor field of valence two g_{ij}, whose discriminant is different from zero, we define in each local space $\mathbf{A}_3(t)$ the scalar product of vectors and require that, under the mapping defining the connection, the scalar product be preserved. The condition of invariance of the scalar product may be written as

$$\nabla g_{ij} = dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = 0. \tag{3}$$

In order that the system of equations (3) be completely integrable, it is necessary and sufficient that $\Omega_i^j = 0$ (the connection is equiaffine).

The space of linear elements of an equiaffine connection, in each local space of which a symmetric scalar product of vectors is defined that is invariant with respect to the mapping, defining the connection, of the neighboring local space onto the initial one, will be called a metric space of linear elements of Euclidean connection.

4. By a curve of the space $\mathfrak{F}_3$ it is natural to understand a one-parameter manifold of linear elements. The centers of this manifold of linear elements form a one-dimensional manifold of points—the centroid of the curve under consideration. Two cases are possible: the linear elements may touch the centroid and may not touch it. The first curve is called centroidal, and the second—noncentroidal ([3]).

If the first curvature of a centroidal curve is equal to zero, then we shall call such a curve a centroidal geodesic.

5. A congruence of centroidal geodesic curves may be defined by the equations

$$\omega_1^\alpha = \lambda_\beta^\alpha \omega^\beta, \qquad d\lambda_\beta^\alpha - \lambda_\gamma^\alpha \omega_\beta^\gamma + \lambda_\beta^\gamma \omega_\gamma^\alpha - \lambda_\beta^\alpha \omega_1^1 = \lambda_{\beta\gamma}^\alpha \omega^\gamma + \lambda_{\beta 1}^\alpha \omega^1, \tag{4}$$

where

$$\lambda_{[\beta\gamma]}^\alpha = R_{1\gamma\beta}^\alpha + S_{1[\gamma|\varepsilon|}\lambda_{\beta]}^\varepsilon + P_{1\varepsilon\delta}\lambda_{[\gamma}^\varepsilon \lambda_{\beta]}^\delta - \lambda_\varepsilon^\alpha (R_{\gamma\beta}^\varepsilon + C_{[\gamma|\delta|}\lambda_{\beta]}^\delta),$$

$$\lambda_{\beta 1}^\alpha = 2R_{11\beta}^\alpha + S_{11\gamma}^\alpha \lambda_\beta^\gamma - 2\lambda_\gamma^\alpha R_{1\beta}^\gamma + \lambda_\varepsilon^\alpha C_{1\gamma}^\varepsilon \lambda_\beta^\gamma - \lambda_\gamma^\alpha \lambda_\beta^\gamma.$$

With the aid of the tensors λ_β^α and g_{ij} we construct the following tensors:

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{1\alpha}g_{1\beta}}{g_{11}}, \qquad \tilde{\lambda}_\beta^\alpha = \frac{\lambda_\beta^\alpha}{F}, \qquad a_{\alpha\beta} = \gamma_{\varepsilon\nu}\tilde{\lambda}_\alpha^\varepsilon \tilde{\lambda}_\beta^\nu,$$

$$b_{\alpha\beta} = \gamma_{\alpha\varepsilon}\tilde{\lambda}_\beta^\varepsilon, \qquad b_{(\alpha\beta)} = \frac{1}{2}\left(b_{\alpha\beta} + b_{\beta\alpha}\right), \tag{5}$$

where $F = \sqrt{g_{11}}$ (it is assumed that the quadratic form $g_{ij}x^i x^j$ is positive definite).

Let g and g' be two infinitely close centroidal geodesics, passing through the points $A(u)$ and $A(u+du)$ of the base of the space $\mathfrak{F}_3$, with unit tangent vectors $\vec{\tau}(u)$ and $\vec{\tau}(u+du)$ $\left(\vec{\tau} = \dfrac{\mathbf{e}_1}{F}\right)$, belonging to the congruence under consideration. Under the mapping defining the connection (1) we obtain

$$\vec{\tau}(u+du) \to \vec{\tau}(u,du) = \vec{\tau}(u) + \frac{\omega_1^\alpha}{F}\left[\mathbf{e}_\alpha(u) - \frac{g_{1\alpha}}{g_{11}}\mathbf{e}_1(u)\right] + \cdots.$$

By the angle $d\varphi$ between two infinitely close "generators" of the congruence we shall mean the principal linear part of the angle between the vectors $\vec{\tau}(u)$ and $\vec{\tau}(u,du)$. It is determined by the formula:

$$d\varphi^2 = a_{\alpha\beta}\omega^\alpha\omega^\beta. \tag{6}$$

If g' runs through a differential neighborhood of g, then the vectors $\vec{\tau}(u, du)$ in the space $A_3(t)$ describe a surface which it is natural to call the local spherical image of the congruence. The quadratic form (6) determines the linear element of this spherical image. The class of congruences of centroidal geodesics of the space $\mathfrak{F}_3$ for which the tensor $a_{\alpha\beta}$ is equal to zero is an analogue of the class of cylindrical congruences of the space E_3.

We shall call a one-parameter family of centroidal geodesics belonging to the congruence under consideration a ruled surface of the congruence.

The development of g in the space $A_3(t)$ is the straight line l passing through $A(u)$ with unit direction vector $\vec{\tau}(u)$. Let us map $A_3 + dA_3$ onto $A_3(t)$. Under the correspondence determining the mapping (1), the development l' of the centroidal geodesic g' is mapped onto the straight line $l(u, du)$, passing through the point $A(u, du)$ with direction vector $\vec{\tau}(u, du)$. The shortest distance between these two straight lines $l(u)$ and $l(u, du)$ we shall call the distance between two infinitely close "generators" of the congruence. The quantity r, determined by the formula

$$r = -\frac{b_{(\alpha\beta)}\omega^\alpha\omega^\beta}{a_{\gamma\varepsilon}\omega^\gamma\omega^\varepsilon}, \tag{7}$$

determines the abscissa of the image of the foot of the common perpendicular of the two centroidal geodesics g and g' on the development of the first of them, which lies in the space $A_3(t)$.

If through g one draws a ruled surface of the congruence so that it contains also g', determined by the ratio $\omega^2 : \omega^3$, then the line of striction of this surface will pass through such a point of the curve g whose image on its development has abscissa r. The null lines of the form $b_{(\alpha\beta)}\omega^\alpha\omega^\beta$ determine those ruled surfaces of the congruence for which the lines of striction pass through the point $A(u)$ of the supporting surface of the congruence.

The class of congruences of centroidal geodesics of the space $\mathfrak{F}_3$ for which

$$b_{(\alpha\beta)} = k a_{\alpha\beta},$$

is an analogue of the class of pseudospherical congruences of the space E_3.

Those points of the centroidal geodesic g whose images on its development l have abscissas equal to the extremal values of the quantity r shall be called boundary points. The abscissas of the images of the boundary points are determined by the equation:

$$ar^2 + \sigma^{\alpha\beta}\sigma^{\varepsilon\gamma}a_{\beta\gamma}b_{(\alpha\varepsilon)}r + b = 0, \tag{8}$$

where

$$a = \det \|a_{\alpha\beta}\|, \qquad b = \det \|b_{(\alpha\beta)}\|, \qquad \sigma^{22} = \sigma^{33} = 0, \qquad \sigma^{23} = -\sigma^{32} = 1.$$

The ruled surfaces of the congruence whose lines of striction intersect the original centroidal geodesic at its boundary points shall be called principal ruled surfaces. They are determined by the differential equation:

$$\sigma^{\alpha\beta} b_{(\alpha\gamma)} a_{\beta\mu} \omega^{\gamma} \omega^{\mu} = 0. \tag{9}$$

Let g and g', passing through $\mathbf{A}(u)$ and $\mathbf{A}(u+du)$, respectively, intersect at the point $\mathbf{F}$. Let us construct developments of the space $\mathfrak{F}_3$ along these centroidal geodesics. We then obtain:

$$\mathbf{F} \xrightarrow{g} \mathbf{F}(u) = \mathbf{A}(u) + \rho(u)\,\vec{\tau}(u),$$

$$\mathbf{F} \xrightarrow{g'} \mathbf{F}(u+du) = \mathbf{A}(u+du) + \rho(u+du)\,\vec{\tau}(u+du).$$

We shall call the point $\mathbf{F}$ the focus of the "generator" g, if, under the defining connectivity of the mapping of the neighboring local space onto the original one, the image of the point $\mathbf{F}(u+du)$ lies on its development. This condition is satisfied if and only if

$$(\delta^{\alpha}_{\beta} + \rho\lambda^{\alpha}_{\beta})\,\omega^{\beta} = 0 \qquad (\delta^{\alpha}_{\beta} \text{ is the Kronecker symbol}).$$

Thus, the roots of the equation

$$K\rho^2 + 2H\rho + 1 = 0, \tag{10}$$

where

$$K = \det \|\tilde{\lambda}^{\alpha}_{\beta}\|, \qquad H = sp\|\tilde{\lambda}^{\alpha}_{\beta}\|,$$

determine the abscissae of the images of the focal points of the centroidal geodesic g. The differential equation of the developable surfaces has the form

$$\sigma_{\alpha\beta}\tilde{\lambda}^{\alpha}_{\gamma}\omega^{\gamma}\omega^{\beta} = 0, \tag{11}$$

where

$$\sigma_{22} = \sigma_{33} = 0, \qquad \sigma_{23} = -\sigma_{32} = 1.$$

In conclusion I take this opportunity to express my deep gratitude to S. P. Finikov for his guidance and assistance in the work.

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Received
9 II 1960

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.