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V. P. LEONOV

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**Abstract**

**Full Text**

*MATHEMATICS*

**V. P. LEONOV**

## THE USE OF THE CHARACTERISTIC FUNCTIONAL AND SEMI-INVARIANTS IN THE ERGODIC THEORY OF STATIONARY PROCESSES

*(Presented by Academician A. N. Kolmogorov, 12 III 1960)*

The present work is a continuation of paper <sup>(1)</sup> and uses its basic definitions and notation:  $\mathfrak{S}_1$ ,  $\chi_\xi(g)$ ,  $m_\xi^{(k)}(t_1, \dots, t_k)$ ,  $s_\xi^{(k)}(t_1, \dots, t_k)$ ,  $F_\xi^{(k)}(\Lambda)$ ,  $S^\infty$ ,  $\Phi^{(\infty)}$ ,  $\Delta^{(\infty)}$  (see the introduction to <sup>(1)</sup>). The results are formulated for processes with continuous time; however, all of them, with obvious changes, carry over also to the case of discrete time.

Let  $\Omega$  be the space of all real-valued functions  $\omega(t)$  of a real variable,  $-\infty < t < \infty$ ; let  $\mathfrak{S}$  be the  $\sigma$ -algebra of its subsets generated by cylinder sets with a finite-dimensional base; let  $\xi(t)$  be a stationary, in the narrow sense, process with continuous time; let  $L^p$  be the space of complex  $\mathfrak{S}$ -measurable functions  $f(\omega)$  such that  $M|f(\xi(t))|^p < \infty$ , and let

$$S^\tau \omega(t) = \omega(t + \tau), \quad U^\tau f(\omega) = f(S^\tau \omega).$$

We shall say that the process  $\xi(t)$  has one of the properties: ergodicity, mixing in the wide sense, mixing of degree  $r$ , if the corresponding translation group of one-to-one measure-preserving point transformations of the measurable space  $(\Omega, \mathfrak{S})$  has this property (see <sup>(2)</sup>, Chap. IX, §1; for the definition of mixing in the wide sense see <sup>(3)</sup>, Def. 11.2; for the definition of mixing of degree  $r$  see <sup>(4)</sup>, §1).

**Definition 1.** We shall call the process  $\xi(t)$  a **process with mixing of degree  $r$  in the wide sense** if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |M\{U^{\alpha_0 \tau} f_0(\xi(t)) \dots U^{\alpha_r \tau} f_r(\xi(t))\} - M\{f_0(\xi(t))\} \dots M\{f_r(\xi(t))\}|^2 d\tau = 0 \quad (1)$$

for all  $f_0, \dots, f_r \in L^{r+1}$  and for all real\*  $\alpha_0, \dots, \alpha_r$ ;  $\alpha_{i_1} \neq \alpha_{i_2}$  for  $i_1 \neq i_2$ .

It is obvious that mixing of degree 1 coincides with mixing, and mixing of degree 1 in the wide sense coincides with mixing in the wide sense, as defined in (3).

Let  $P_\xi(d\omega)$  be the probability measure on  $(\Omega, \mathfrak{G})$  corresponding to the random process  $\xi(t)$ .

**Definition 2.** We shall say that the process  $\xi(t)$  belongs to

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\* In the case of discrete time, for all integer  $\alpha_0, \dots, \alpha_r; \alpha_{i_1} \neq \alpha_{i_2}$  for  $i_1 \neq i_2$ .  
class  $\overline{S^{(\infty)}} (\xi(t) \in \overline{S^{(\infty)}})$ , if:

- 1)  $\xi(t) \in S^{(\infty)}$ ;
- 2) for any random process  $\eta(t)$  such that

$$m_\eta^{(k)}(t_1, \dots, t_k) = m_\xi^{(k)}(t_1, \dots, t_k)$$

for all  $k \geq 1, t_1, \dots, t_k$ , the equality  $P_\xi(A) = P_\eta(A)$  holds for all  $A \in \mathfrak{G}$ .  
Obviously, if  $\xi(t) \in \overline{S^{(\infty)}}$ , then  $\xi(t)$  is stationary in the narrow sense.

For  $g \in \mathfrak{G}_1$ , denote

$$V^\tau g(dt) = g[d(t - \tau)].$$

**Theorem 1.** For the ergodicity of  $\xi(t)$  it is necessary and sufficient that the relation

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \chi_\xi(g_0 + V^\tau g_1) d\tau = \chi_\xi(g_0) - \chi_\xi(g_1) \quad \text{for all } g_0, g_1 \in \mathfrak{G}_1 \quad (2)$$

be fulfilled.

In order that  $\xi(t)$  be a mixing process of degree  $r$  in the broad sense, it is necessary and sufficient that the relation

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\chi_\xi(V^{\alpha_0 \tau} g_0 + \dots + V^{\alpha_r \tau} g_r) - \chi_\xi(g_0) \dots \chi_\xi(g_r)|^2 d\tau = 0 \quad (3)$$

hold for all  $g_0, \dots, g_r \in \mathfrak{G}_1$  and for all real  $\alpha_0, \dots, \alpha_r; \alpha_{i_1} \neq \alpha_{i_2}$  when  $i_1 \neq i_2$ .\*

In order that  $\xi(t)$  be a mixing process of degree  $r$ , it is necessary and sufficient that the relation

$$\lim_{\min_{0 \leq i < j \leq r} |\tau_i - \tau_j| \rightarrow \infty} \chi_\xi(V^{\tau_0} g_0 + \dots + V^{\tau_r} g_r) = \chi_\xi(g_0) \dots \chi_\xi(g_r). \quad (4)$$

for all  $g_0, \dots, g_r \in \mathfrak{G}_1$ .

**Remark 1.** The theorem remains valid if, in its formulation, the space  $\mathfrak{G}_1$  is replaced by the space of generalized measures concentrated at a finite number

of points; in this case, in (2), (3), (4) the characteristic functionals become finite-dimensional characteristic functions.

It is well known<sup>(5-7)</sup> that if  $\xi(t)$  is a Gaussian process with covariance function  $B_\xi(\tau) = s_\xi^{(2)}(t, t + \tau)$  and spectral function

$$F_\xi(\lambda) = F_\xi^{(2)}((-\infty, \lambda)X(-\infty, \lambda)),$$

then, in order that  $\xi(t)$  be a mixing process in the broad sense, it is necessary and sufficient that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |B_\xi(\tau)|^2 d\tau = 0,$$

or, equivalently, that  $F_\xi(\lambda)$  be continuous; and, in order that  $\xi(t)$  be a mixing process, it is necessary and sufficient that

$$\lim_{T \rightarrow \infty} B_\xi(\tau) = 0.$$

A direct generalization of these facts is given by the following theorems<sup>\*\*</sup>.

**Theorem 2.** *In order that a process  $\xi(t) \in \overline{S^{(\infty)}}$  be a mixing process of degree  $r$  in the broad sense, it is necessary and sufficient that the relation<sup>\*\*\*</sup>*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left| s_\xi^{(k)}(t_{01} + \alpha_0\tau, \dots, t_{0l_0} + \alpha_0\tau, t_{11} + \alpha_1\tau, \dots, t_{1l_1} + \alpha_1\tau, \dots, \dots, t_{r1} + \alpha_r\tau, \dots, t_{rl_r} + \alpha_r\tau) \right|^2 d\tau = 0 \quad (5)$$

\* See the footnote to Definition 1.

\*\* The statements given above concerning Gaussian processes are contained in Theorems 2, 3, 5, since in this case  $\xi(t) \in \overline{S^{(\infty)}} \cap \Phi^{(\infty)}$ .

\*\*\* By virtue of the symmetry of  $s_\xi^{(k)}(t_1, \dots, t_k)$ , from the fulfillment of relation (5) follows

the fulfillment of the corresponding relations under arbitrary permutations of the arguments of  $s_\xi^{(k)}$ .

for all  $k \geq 2$ ,  $0 < l_0 < k$ ,  $l_i \geq 0$ ,  $l_0 + l_1 + \dots + l_r = k$ ,  $t_{ij}^*$ ,  $0 \leq i \leq r$ ,  $1 \leq j \leq l_i$ , real<sup>\*\*</sup>  $\alpha_0, \dots, \alpha_r$ ;  $\alpha_{i_1} \neq \alpha_{i_2}$  for  $i_1 \neq i_2$ .

Denote by  $L_k^l(\nu)$  the hyperplane of the space  $R^k$  of the variables  $\lambda_1, \dots, \lambda_k$ , defined by the relation  $\lambda_1 + \dots + \lambda_l = \nu$ .

**Theorem 3.** In order that the process  $\xi(t) \in \overline{S}^{(\infty)} \cap \Phi^{(\infty)}$  be a process with mixing in the broad sense, it is necessary and sufficient that the relation\*\*\*

$$|F_{\xi}^k|(L_k^l(\nu)) = 0 \quad (6)$$

hold for all  $k \geq 2$ ,  $0 < l < k$ ,  $-\infty < \nu < \infty$ , where  $|F_{\xi}^{(k)}|(\Lambda)$  is the total variation of the complex measure  $F_{\xi}^{(k)}(\Lambda)$ .

**Theorem 4.** In order that the process  $\xi(t) \in \overline{S}^{(\infty)} \cap \Phi^{(\infty)}$  be a process with mixing of all degrees in the broad sense, it is necessary and sufficient that the relation

$$|F_{\xi}^{(k)}|(L) = 0 \quad (7)$$

hold for all  $k \geq 2$  and for all hyperplanes\*\*\*\*  $L$  of the space  $R^k$ , distinct from  $L_k^k(0)$ .

**Remark 2.** Since for  $\xi(t) \in \overline{S}^{(\infty)} \cap \Phi^{(\infty)}$  the measure  $F_{\xi}^{(k)}(\Lambda)$  is concentrated on  $L_k^k(0)$  (see (1)), Theorem 4 may be formulated as follows:

In order that the process  $\xi(t) \in \overline{S}^{(\infty)} \cap \Phi^{(\infty)}$  be a process with mixing of all degrees in the broad sense, it is necessary and sufficient that relation (7) hold for all  $k \geq 2$  and for all hyperplanes  $L$  of the subspace  $L_k^k(0)$ .

**Theorem 5.** In order that the process  $\xi(t) \in \overline{S}^{(\infty)}$  be a process with mixing of degree  $r$ , it is necessary and sufficient that the relation\*\*\*\*\*

$$\lim_{\min_{0 \leq i < j \leq r} |\tau_i - \tau_j| \rightarrow \infty} s_{\xi}^{(k)}(t_{01} + \tau_0, \dots, t_{0l_0} + \tau_0, t_{11} + \tau_1, \dots, t_{1l_1} + \tau_1, \dots, \dots, t_{r1} + \tau_r, \dots, t_{rl_r} + \tau_r) = 0 \quad (8)$$

hold for all  $k \geq 2$ ,  $0 < l_0 < k$ ,  $l_i \geq 0$ ,  $l_0 + l_1 + \dots + l_r = k$ ,  $t_{ij}^*$ ,  $0 \leq i \leq r$ ,  $1 \leq j \leq l_i$ .

**Theorem 6.** In order that a stochastically continuous process  $\xi(t) \in \overline{S}^{(\infty)}$  be a process with mixing of all degrees, it is necessary and sufficient that the relation

$$\lim_{\max_{i,j} |t_i - t_j| \rightarrow \infty} s_{\xi}^{(k)}(t_1, \dots, t_k) = 0 \quad \text{for all } k \geq 2. \quad (9)$$

From the theorems stated there follow:

**Corollary 1.** If  $\xi(t) \in \overline{S}^{(\infty)} \cap \Delta^{(\infty)}$ , then  $\xi(t)$  is a process with mixing of all degrees.

**Corollary 2.** For Gaussian processes, mixing of all degrees coincides with mixing, and mixing of all degrees in the broad sense coincides with mixing in the broad sense.

\* Including also such sets  $t_{ij}$  in which not all  $t_{ij}$  are distinct.

\*\* See the footnote to Definition 1.

\*\*\* By virtue of the symmetry of the measure  $F_\xi^{(k)}(\Lambda)$  (see the introduction to (1)), it follows from relation (6) that the corresponding relations are satisfied for all hyperplanes of the form  $\lambda_{i_1} + \dots + \lambda_{i_l} = \nu$ .

\*\*\*\* In the discrete-time case, for all hyperplanes of the form  $\alpha_1 \lambda_1 + \dots + \alpha_k \lambda_k = \nu$  with integer  $\alpha_i$ , distinct from  $L_k^k(0)$ .

\*\*\*\*\* See the footnote to Theorem 2.

Let us note that the condition  $\xi(t) \in \overline{S^{(\infty)}}$  contained in the formulations of Theorems 2-6 is necessary if we wish to decide the question of the ergodic properties of the process  $\xi(t)$  on the basis only of its moments (or semi-invariants), namely:

*If  $\xi(t) \in S^{(\infty)} \setminus \overline{S^{(\infty)}}$  and is stationary in the narrow sense, then there exists a continuum of nonergodic processes, all of whose moments coincide with the moments of the process  $\xi(t)$ .*

A proper strengthening of condition (9) leads to the asymptotic normality of the integral

$$\zeta_T = \int_0^T \xi(t) dt,$$

i.e., if we denote

$$b_T = D \left( \int_0^T \xi(t) dt \right) = \int_0^T \int_0^T B_\xi(t_1 - t_2) dt_1 dt_2,$$

then the following holds:

**Theorem 7.** If  $\xi(t) \in S^{(\infty)}$  and

$$\int_0^T \dots \int_0^T S_\xi^{(k)}(t_1, \dots, t_k) dt_1 \dots dt_k = o(b_T^{k/2}) \quad \text{for all } k \geq 3, \quad (10)$$

then

$$\lim_{T \rightarrow \infty} P_\xi \left\{ \frac{\zeta_T - m_\xi^{(1)} T}{\sqrt{b_T}} < y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-z^2/2} dz. \quad (11)$$

In essence, Theorem 7 is of interest only in the case when  $b_T \rightarrow \infty$  as  $T \rightarrow \infty$ . In this connection we make the following remarks.

**Remark 3.** If

$$\lim_{\lambda \rightarrow 0} \frac{F_\xi(\lambda) - F_\xi(-\lambda)}{(2\lambda)^\gamma} = f_\gamma, \quad 0 \leq \gamma < 2,$$

$$\lim_{T \rightarrow \infty} \frac{b_T}{T^{2-\gamma}} \geq \frac{4}{\pi^2} (2\pi)^\gamma f_\gamma.$$

**Remark 4.** If

$$F_\xi(\lambda) = \int_{-\infty}^{\lambda} f_\xi(\nu) d\nu, \quad f_\xi(\lambda) = \lambda^\beta f_1(\lambda), \quad -1 < \beta < 1,$$

and  $f_1(\lambda)$  is continuous at the point  $\lambda = 0$ , then

$$b_T = d_\beta f_1(0) T^{1-\beta} + o(T^{1-\beta}), \quad \text{where} \quad d_\beta = 2 \int_{-\infty}^{\infty} \frac{1 - \cos \nu}{\nu^2 - \beta} d\nu.$$

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Mathematical Institute named after V. A. Steklov  
Academy of Sciences of the USSR

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*Note: Figure translations are in progress. See original paper for figures.*

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