MATHEMATICS

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1960

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Abstract

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MATHEMATICS

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A SUPPLEMENT TO S. M. NIKOLSKII' S EMBEDDING THEOREM FOR THE CLASS $H_{p_1,\dots,p_n}^{*(r_1,\dots,r_n)}$ AND THE IMPOSSIBILITY OF IMPROVING ONE ESTIMATE

(Presented by Academician I. M. Vinogradov, 7 VII 1960)

In this note an embedding theorem is proved for the classes $H_{p_1,\dots,p_n}^{*(r_1,\dots,r_n)}$ by S. M. Nikolskii' s method. In addition, it is proved that one of his estimates for the classes $H_p^{(r_1,\dots,r_n)}$ cannot be improved.

S. M. Nikolskii proved in ([6]) an embedding theorem for the class $H_{p_1,\dots,p_n}^{(r_1,\dots,r_n)}$ in the case when $G = R_n$ is the whole n-dimensional space and when $p_i \le p$ $(i = 1, 2, \dots, n)$. We consider, by the same methods, the more general case when

$$1 \le p_1 \le p_2 \le \dots \le p_l \le p \le p_{l+1} \le \dots \le p_n \le \infty \qquad (l = 1, 2, \dots, n)$$

for the class $H_{p_1,\dots,p_n}^{*(r_1,\dots,r_n)}$ (periodic case).

The proof in this case is again carried out on the basis of approximations of a function f by entire functions of finite degree. In doing so we use the generalized Jackson theorem, proved by S. M. Nikolskii ([6]) as applied to the classes $H_{p_1,\dots,p_n}^{*(r_1,\dots,r_n)}$. In addition, two inequalities for trigonometric polynomials $T = T_{\nu_1,\dots,\nu_n}$ of orders $\nu_1, \dots, \nu_n$, respectively in the variables $x_1, \dots, x_n$, are an essential tool in the proof:

$$\|T\|_{L_{p'}^{(n)}} \le 2^n \left(\prod_1^n \nu_i\right)^{1/p-1/p'} \|T\|_{L_p^{(n)}} \qquad (1 \le p < p' \le \infty), \tag{1}$$

$$\|T\|_{L_p^{(m)}} \le 2^n \left(\prod_{m+1}^n \nu_i\right)^{1/p} \|T\|_{L_p^{(n)}} \qquad (1 \le p \le \infty;\ 1 \le m \le n), \tag{2}$$

where

$$\|f\|_{L_p^{(m)}} = \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |f(x_1, \dots, x_m, x_{m+1}, \dots, x_n)|^p \, dx_1 \cdots dx_m \right)^{1/p}.$$

These inequalities were obtained in the works of S. M. Nikolskii ([3,4]). Inequality (1), for $n = 1$ and $p' = \infty$, becomes an inequality obtained by Jackson ([1]).

In addition, the following lemma is used, generalizing the inverse approximation theorem of Bernstein:

Lemma 1. Let $r > 0$, $f \in L_p^{(n)}$, and suppose that for some sequence $T_{\nu,\infty,\dots,\infty}$, for all ν running through the geometric progression $\nu = a^s$ ($s = 0, 1, 2, \dots;\ a > 1$), the inequality

$$\|f - T_{\nu,\infty,\dots,\infty}\|_{L_p^{(n)}} < \frac{K}{\nu^r} \tag{3}$$

holds.

Then

$$f \in H_{px_1}^{*(r)}(M),$$

where

$$M < C \left(\|f\|_{L_p^{(n)}} + K \right), \tag{4}$$

where the constant C does not depend on the multiplier standing next to it.

For the proof see paper ([4]), Theorem 8 (the formulation and the remark on p. 264 at the end of the proof of the theorem).

We have proved the following theorem:

Theorem 1. Let, for the numbers considered below, the inequalities hold:

$$r_i > 0;$$

$$1 \le p_1 \le \dots \le p_l \le p \le p_{l+1} \le \dots \le p_n \le \infty;$$

n, m, l are natural numbers, and

$$1 \le m \le n, \qquad 1 \le l \le n.$$

Moreover:

a) when $l \le m \le n$

$$\rho_i = \frac{r_i\left[1-\sum_1^l\left(\frac{1}{p_d}-\frac{1}{p}\right)\frac{1}{r_d}\right]\left[1-\frac{1}{p}\sum_{m+1}^n\frac{1}{r_d}\right]}{1+\left(\frac{1}{p_i}-\frac{1}{p}\right)\sum_1^n\frac{1}{r_d}-\sum_1^l\left(\frac{1}{p_d}-\frac{1}{p}\right)\frac{1}{r_d}} > 0, \quad i = 1, 2, \dots, l; \qquad (5)$$

$$\rho_i = r_i\left(1-\frac{1}{p}\sum_{m+1}^n\frac{1}{r_d}\right) > 0, \qquad i = l+1, \dots, n;$$

b) when $m < l \le n$

$$\rho_i = r_i\frac{\left(1-\frac{1}{p}\sum_{l+1}^n\frac{1}{r_d}\right)\left[1-\sum_1^m\left(\frac{1}{p_d}-\frac{1}{p}\right)\frac{1}{r_d}\right]}{1-\sum_1^l\frac{1}{r_d}\left(\frac{1}{p_d}-\frac{1}{p}\right)+\left(\frac{1}{p_i}-\frac{1}{p}\right)\sum_{l+1}^n\frac{1}{r_d}} -$$

$$-r_i\frac{\left(1+\frac{1}{p}\sum_1^m\frac{1}{r_d}\right)\sum_{m+1}^l\frac{1}{p_d r_d}-\frac{1}{p}\sum_1^m\frac{1}{p_d r_d}\sum_{m+1}^l\frac{1}{r_d}}{1-\sum_1^l\frac{1}{r_d}\left(\frac{1}{p_d}-\frac{1}{p}\right)+\left(\frac{1}{p_i}-\frac{1}{p}\right)\sum_{l+1}^n\frac{1}{r_d}} > 0, \quad i = 1, 2, \dots, l; \ (6)$$

$$\rho_i = r_i\frac{\left(1-\frac{1}{p}\sum_{l+1}^n\frac{1}{r_d}\right)\left[1-\sum_1^m\left(\frac{1}{r_d}-\frac{1}{p}\right)\frac{1}{r_d}\right]}{1-\sum_1^l\frac{1}{r_d}\left(\frac{1}{p_d}-\frac{1}{p}\right)} -$$

$$-r_i\frac{\left(1+\frac{1}{p}\sum_1^m\frac{1}{r_d}\right)\sum_{m+1}^l\frac{1}{p_d r_d}-\frac{1}{p}\sum_1^m\frac{1}{p_d r_d}\sum_{m+1}^l\frac{1}{r_d}}{1-\sum_1^l\frac{1}{r_d}\left(\frac{1}{p_d}-\frac{1}{p}\right)} > 0, \quad i = l+1, \dots, n.$$

Let, further, a function $f(x_1, \dots, x_n)$, defined in the n-dimensional space R_n, belong to the class

$$H^{*(r_1,\dots,r_n)}_{p_1,\dots,p_n}(M).$$

Then, for any fixed $(x_{m+1}, \dots, x_n)$, the function f, as a function of $x_1, \dots, x_m$, belongs to the class $H_p^{*(\rho_1,\dots,\rho_m)}(\overline{M})$. Moreover, the inequality

$$\|f\|_{L_p^{(m)}} + \overline{M} < C\left(\min_{1\le i\le n}\|f\|_{L_{p_i}^{(n)}} + M\right), \qquad (7)$$

holds, where the constant C does not depend on the set standing next to it.

Remark. For $l = n$ one obtains the theorem of S. M. Nikol' skii [6] for the class $H^{*(r_1,\dots,r_n)}_{p_1,\dots,p_n}(M)$.

A function f belonging to $L_p^{(n)}$ is defined in R_n up to a set of measure zero; therefore, in order to be able to speak of its values on hyperplanes of smaller dimension, the following definition is introduced [5,8]: a function $\varphi(x_1, \dots, x_m)$, where $0 < m < n$, is called the value of the function f in the sense of convergence in the mean (for fixed p, $1 \leq p \leq \infty$) on the m-dimensional hyperplane $x_{m+1} = x_{m+1}^{(0)}, \dots, x_n = x_n^{(0)}$, if the function f can be modified on a set of measure zero so that the limit

$$\lim_{\sum_{i=m+1}^{n} h_i^2 \to 0} \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| f(x_1, \dots, x_m, x_{m+1}^{(0)} + h_{m+1}, \dots, x_n^{(0)} + h_n) - \varphi(x_1, \dots, x_m) \right|^p \, dx_1 \cdots dx_m \right\}^{1/p} = 0 \tag{8}$$

exists.

The boundary value of the function f defined in this way is unique up to the class of equivalent functions with respect to m-dimensional measure; we shall denote it by

$$f(x_1, \dots, x_m, x_{m+1}^{(0)}, \dots, x_n^{(0)}).$$

S. M. Nikol' skii found the order with which the convergence to zero in (8) takes place (see [2]). We formulate this result here in the case when $m = n - 1$.

Let $f \in H_p^{(r_1, \dots, r_n)}(M)$, $\beta_1 = r_1 - 1/p$, where $0 < \beta_1 \leq 1$, and let some values of x_1, h be fixed. Then there exist constants $C_1 > 0$ and $C_2 > 0$, independent of M, f, x_1, h, such that:

for $\beta_1 < 1$,

$$\|\Delta_{x_1}^{(1)}(f; h)\|_{L_p^{(m)}} = \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1 + h, x_2, \dots, x_n) - f(x_1, \dots, x_n)|^p \, dx_2 \cdots dx_n \right\}^{1/p} \leq (C_1 M + C_2 \|f\|_{L_p^{(n)}}) |h|^{\beta_1}; \tag{9}$$

for $\beta_1 = 1$,

$$\|\Delta_{x_1}^{(2)}(f; h)\|_{L_p^{(m)}} =$$

$$= \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1 + h, x_2, \dots, x_n) - 2f(x_1, \dots, x_n) + f(x_1 - h, x_2, \dots, x_n)|^p \, dx_2 \cdots dx_n \right\}^{1/p} \leq (C_1 M + C_2 \|f\|_{L \dots} \tag{10}$$

We have proved that this estimate cannot be improved. In doing so we used a lemma proved by us in (7):

Lemma 2. The function

$$F_N(x) = \left(\frac{\sin\frac{1}{2}Nx}{x}\right)^2$$

has the following properties:

$$|F_N^{(k)}(x)| > CN^{k+3}x, \qquad k = 0, 1, 2, \dots, \tag{11}$$

for $0 < Nx \le \pi/5$ $(N \ge 1)$ and

$$\left\|F_N^{(k)}(x)\right\|_{L_p^{(1)}} = K_p N^{2-1/p}, \qquad p \ge 1.$$

Our theorem is formulated as follows:

Theorem 2. The function

$$f(x_1, \dots, x_n) = \sum_{\nu=1}^{\infty} \frac{\prod_{j=1}^{n} F_{\frac{a^{r_1 \dots r_n}}{r_j}}(x_j)}{r_1 \dots r_n \sqrt[a]{1 + \left(1 + \frac{1}{q}\right)\sum_{i=1}^{n} \frac{1}{r_i}}}, \tag{12}$$

where convergence of the series is understood in the sense of $L_p^{(n)}$, and where

$$F_N(x) = \left(\frac{\sin\frac{1}{2}x}{x}\right)^2, \qquad 1 \le p \le \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

for any $a > 1$ belongs to the class $H_p^{(r_1,\dots,r_n)}$ (9).

Moreover, for sufficiently large $a > 1$ one can find for it a sequence $h \to 0$ $(h > 0)$ such that, for some number $K > 0$ independent of h, and for $x_1 = 0$, the inequalities hold:

$$\left\|\Delta_{x_1=0}^{(1)}(f; h)\right\|_{L_p^{(n-1)}} > Kh^{\beta_1} \quad \text{for } 0 < \beta_1 < 1; \tag{13}$$

$$\left\|\Delta_{x_1=0}^{2}(f; h)\right\|_{L_p^{(n-1)}} > Kh \quad \text{for } \beta_1 = 1. \tag{14}$$

This theorem proves that the above-formulated theorem of S. M. Nikol' skii cannot be improved.

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Received
29 VI 1960

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Note: Figure translations are in progress. See original paper for figures.