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Abstract

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THEOREMS ON THE EXISTENCE AND EMBEDDING OF SUBGROUPS IN A FINITE GROUP

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§ 1. In the present paper we consider questions of the existence and embedding of subgroups in a finite group in connection with the notions introduced by S. A. Chunikhin in (^{1,2}): those of a block, separable, and reduced Π-block divisor of the order of a group, and with Theorem 14 obtained by him in (^{1}).

The terminology and notation are taken from works ($^{1-3}$).

§ 2. **Lemma 1.** *Let s be a separable divisor of the order g of a group G, and let A be a subgroup of order a of the group G. Then (s, a) is a separable divisor of the order a of the subgroup A.*

Lemma 2. *If N is an invariant subgroup of order n of a group G, and the number s is a separable divisor of the order g of the group G, then the number*

$$\frac{s}{(s, n)} = d$$

is a separable divisor of the order $\frac{g}{n}$ of the factor group G/N.

Theorem 1. *Let h' be a reduced Π-block divisor, and s a separable divisor of the order g of a group G. Then the group G has at least one subgroup of order*

$$\frac{h' s}{(h', s)}\, c,$$

where c is a Π-prime number.

§ 3. **Definition.** A divisor hs of the order g of a group G will be called a **block-separable divisor** if it is the product of a block divisor h and a separable divisor s of the order g of the group G.

Lemma 3. *If hs is a block-separable divisor of the order g of a group G, and if a is the order of a subgroup A of the group G, then (hs, a) is a block-separable divisor of the order a of the subgroup A.*

Lemma 4. *Let N be an invariant subgroup of order n of a group G, and let the number hs be a block-separable divisor of the order g of the group G. Then the number*

$$\frac{hs}{(hs, n)}$$

is a block-separable divisor of the order $\frac{g}{n}$ of the factor group G/N.

Theorem 2. *If a divisor hs of the order g of a group G is block-separable, then every soluble subgroup A of order a dividing hs is contained in at least one of the subgroups of order hs of the group G.*

§ 4. **Lemma 5.** *If h is a block divisor and s is a separable divisor of the order g of a group G, then their greatest common divisor (h, s) is a separable divisor of the order g of the group G.*

Theorem 3. Let

$$h_1, h_2, \dots, h_n \tag{1}$$

be some nonempty set of block divisors, and

$$s_1, s_2, \dots, s_m \tag{2}$$

some nonempty set of separable divisors of the order g of the group G. Then for any two nonempty subsets $h_{i_1}, h_{i_2}, \dots, h_{i_r}$ and

$h_{i_1}, h_{i_2}, \dots, h_{i_l}$ of the set (1), in the group G there exist at least one subgroup of each of the orders:

$$A = (h_{j_1}, s_{k_1}) \left(\frac{h_{j_2}}{(h_{j_2}, h_{j_1})}, s_{k_2} \right) \cdots \left(\frac{h_{j_l}}{(h_{j_l}, h_{j_1} h_{j_2} \cdots h_{j_{l-1}})}, s_{k_l} \right),$$

$$B = \frac{(g, h_{i_1} h_{i_2} \cdots h_{i_r}) \cdot A}{((g, h_{i_1} h_{i_2} \cdots h_{i_r}), A)},$$

where each of the numbers $s_{k_1}, s_{k_2}, \dots, s_{k_l}$ is equal to one of the proper divisors of the set (2), and all subgroups of each of the orders A are conjugate to one another in the group G, while those subgroups of order B for which

$$\frac{A}{((g, h_{i_1} h_{i_2} \cdots h_{i_r}), A)} = d > 1$$

are $\Pi(d)$-solvable.

Theorem 4. If h_1 and h_2 are relatively prime divisors, and s is a proper divisor of the order g of the group G, then G has at least one subgroup of each of the orders:

$$a_1 = \frac{h_1}{d}(d, s), \qquad a_2 = \frac{h_2}{d}(d, s), \qquad a_3 = \frac{d(h_1, s)}{(d, s)},$$

$$a_4 = \frac{d(h_2, s)}{(d, s)},$$

$$a_5 = \frac{h_1(h_2, s)}{(d, s)}, \qquad a_6 = \frac{h_2(h_1, s)}{(d, s)}, \qquad a_7 = \frac{h_1}{d}(h_2, s),$$

$$a_8 = \frac{h_2}{d}(h_1, s), \qquad a_9 = \frac{h_1 h_2}{d^2}(d, s), \qquad a_{10} = d\left(\frac{h_1 h_2}{d^2}, s\right),$$

where $d = (h_1, h_2)$, and:

if $(d, s) = m_1 > 1$, subgroups of orders a_1, a_2, a_9 are $\Pi(m_1)$-solvable;

if $\dfrac{(h_1, s)}{(d, s)} = m_2 > 1$, subgroups of orders a_3, a_6 are $\Pi(m_2)$-solvable;

if $\dfrac{(h_2, s)}{(d, s)} = m_3 > 1$, subgroups of orders a_4, a_5 are $\Pi(m_3)$-solvable;

if $(h_2, s) = m_4 > 1$, a subgroup of order a_7 is $\Pi(m_4)$-solvable;

if $(h_1, s) = m_5 > 1$, a subgroup of order a_8 is $\Pi(m_5)$-solvable;

if $\left(\dfrac{h_1 h_2}{d^2}, s\right) = m_6 > 1$, a subgroup of order a_{10} is $\Pi(m_6)$-solvable.

Theorem 5. If

$$h'_1, h'_2, \dots, h'_n \tag{3}$$

is the set of reduced Π-relatively prime divisors, and

$$s_1, s_2, \dots, s_m \tag{4}$$

is the set of proper divisors of the order g of the group G, then for any two nonempty subsets $h'_{i_1}, h'_{i_2}, \dots, h'_{i_r}$ and $h'_{j_1}, h'_{j_2}, \dots, h'_{j_l}$ of the set (3), in the group G there exists at least one subgroup of each of the orders:

$$A = (h'_{j_1}, s_{k_1}) \left(\frac{h'_{j_2}}{(h'_{j_2}, h'_{j_1})}, s_{k_2}\right) \cdots \left(\frac{h'_{j_l}}{(h'_{j_l}, h'_{j_1} h'_{j_2} \cdots h'_{j_{l-1}})}, s_{k_l}\right),$$

$$B = \frac{(g, h'_{i_1} h'_{i_2} \cdots h'_{i_r}) \cdot A}{((g, h'_{i_1} h'_{i_2} \cdots h'_{i_r}), A)}\, c,$$

where each of the numbers $s_{k_1}, s_{k_2}, \dots, s_{k_l}$ is equal to one of the numbers of the set (4), and c is a Π-prime number; moreover, all subgroups of each of the orders A are conjugate to one another in the group G.

§ 5. In conclusion, we note that from Theorem 2, for $s = 1$, there follows Theorem 5 of paper (^{1}), and for $h = 1$—the theorem on the embedding of subgroups of a Π-separable group, D5.2 of paper (^{5}).

If in Theorem 1 the set Π contains all distinct prime divisors of the order of the group, then from it one obtains Theorem 14 of paper (^{1}). As is known, from the 14th theorem of S. A. Chunikhin there follow, as its special cases, the theorems on the existence of subgroups: those of Sylow, P. Hall (for solvable groups), Schur (on factorization of groups), S. A. Chunikhin (Theorem 1 of (^{1})) and the fundamental theorem on Π-separable groups (^{3}).

Theorem 3 is a special case of Theorem 5, when the set Π contains all distinct prime divisors of the order of the group.

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REFERENCES

[1] S. A. Chunikhin, *Matem. sborn.*, **39** (81), No. 3, 465 (1956).
[2] S. A. Chunikhin, *Matem. sborn.*, **43** (85), No. 1, 49 (1957).
[3] S. A. Chunikhin, *DAN*, **59**, No. 3, 443 (1948).
[4] S. A. Chunikhin, *DAN*, **119**, No. 5, 888 (1958).
[5] P. Hall, *Proc. Lond. Math. Soc.*, (3), **6**, No. 22, 286 (1956).
[6] O. Yu. Schmidt, *Abstract Theory of Groups*, 1933.

Note: Figure translations are in progress. See original paper for figures.