MATHEMATICS

1960

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Abstract

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MATHEMATICS

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ON STRONGLY ELLIPTIC MONGE—AMPÈRE EQUATIONS

(Presented by Academician V. I. Smirnov on 14 XII 1959)

The Monge—Ampère equation

$$\vartheta'(rt - s^2) = ar + 2bs + ct + \varphi \qquad (*)$$

with continuous coefficients depending on x, y, z, p, q, is called strongly elliptic if $\vartheta' > 0$, $\varphi > 0$, and the quadratic form in the variables ξ, η

$$a\xi^2 + 2b\xi\eta + c\eta^2$$

is nonnegative ([1]). For such equations the notion of a conditional (or generalized) solution is introduced as follows.

Let a regular function $z(x, y)$ in Cartesian coordinates x, y, z define a convex surface $F : z = z(x, y)$, whose convexity is directed toward the side $z < 0$. Define on the xy-plane three set functions:

$$\vartheta_F(H) = \iint_H \vartheta'(rt - s^2)\, dx\, dy,$$

$$h_F(H) = \iint_H (ar + 2bs + ct)\, dx\, dy,$$

$$\sigma_F(H) = \iint_H \varphi\, dx\, dy.$$

We shall call them, respectively, the conditional total curvature, the conditional mean curvature, and the conditional area of the surface. By passage to the limit in the sense of weak convergence, the set functions ϑ, h, and σ are defined for any convex surface. Each of them is a nonnegative completely additive function on the ring of Borel sets.

Now a conditional solution of equation $(*)$ can be defined as a function $z(x, y)$ that defines a convex surface F, whose convexity is directed toward the side $z < 0$, satisfying the condition

$$\vartheta_F = h_F + \sigma_F.$$

Under certain assumptions concerning the coefficients of equation $(*)$, I. Ya. Bakelman in ([1]) proved the existence of conditional solutions of equation $(*)$ and the solvability of the Dirichlet problem for convex domains in a certain generalized formulation.

In order that the Dirichlet problem in the usual sense for conditional solutions be solvable, the coefficients of equation $(*)$ must be subjected to additional restrictions. If one assumes a certain order of growth (decrease) of the coefficients of equation $(*)$ as $p^2 + q^2 \to \infty$, then the solvability condition for the Dirichlet problem for equation $(*)$ can be formulated comparatively simply. Namely, the following theorem holds:

Theorem 1. Let the curve bounding a convex domain G of the xy-plane have positive curvature*. The Dirichlet problem for the strongly elliptic Monge–Ampère equation

$$rt - s^2 = ar + 2bs + ct + \varphi \tag{**}$$

in the domain G, for arbitrary continuous boundary values, is solvable if the following condition is satisfied: as $p^2 + q^2 \to \infty$,

$$a,\ |b|,\ c < N\,(p^2+q^2)^{\frac{1}{2}-\alpha}, \qquad \varphi < N\,(p^2+q^2),$$

where $N = \text{const} < \infty,\ \alpha = \text{const} > 0$.

Theorem 1a. Theorem 1 also holds for $\alpha = 0$, if the curvature of the curve bounding the domain G is sufficiently large (and, consequently, the domain itself is sufficiently small).

For conditional solutions of the strongly elliptic Monge–Ampère equation the following uniqueness theorem holds:

Theorem 2. Let the coefficients of equation $(**)$ be differentiable with respect to each of the variables, and let the coefficient φ and the quadratic form $a\xi^2 + 2b\xi\eta + c\eta^2$, as functions of z, be nondecreasing. Then any two conditional solutions of equation $(**)$ that coincide on the boundary of the domain G coincide identically in G.

A conditional solution may be nonregular even in the sense of twice differentiability. In some cases, given sufficient regularity of the coefficients of the equation, regularity of conditional solutions can be guaranteed. For example, the following theorem holds:

Theorem 3. Let the coefficients of the strongly elliptic Monge–Ampère equation $(**)$ be regular (k times differentiable, $k \geqslant 3$) and satisfy the conditions:

1) the function φ is nondecreasing in z and convex in p, q;
2) the quadratic form $a\xi^2 + 2b\xi\eta + c\eta^2$ is nondecreasing in z and convex in p, q.

Then every conditional solution of equation $(**)$ is regular ($k+1$ times differentiable). If, moreover, the coefficients of the equation are analytic, then the solution is analytic.

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Received
11 XII 1959

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1. I. Ya. Bakelman, DAN, **126**, No. 5 (1959).

* This means that for any two sufficiently close points of the curve, the ratio of the angle between the supporting lines at these points to the distance between the points is bounded below by a positive number.

Note: Figure translations are in progress. See original paper for figures.