PROJECTIVE AND PROJECTIVE-METRIC TRANSPORTS IN MANIFOLDS WITH AFFINE CONNECTION AND IN RIEMANNIAN SPACES

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Abstract

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MATHEMATICS

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PROJECTIVE AND PROJECTIVE-METRIC TRANSPORTS IN MANIFOLDS WITH AFFINE CONNECTION AND IN RIEMANNIAN SPACES

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1. Let an object (a_i) be given on a differentiable manifold $\{V^n\}$, whose components, under a transformation $x^{i'} = x^{i'}(x^i)$ of local coordinate systems of the manifold, transform according to the law

$$a_{i'} = \frac{\partial x^i}{\partial x^{i'}} a_i - \frac{1}{(n+1)} \frac{\partial \ln \det \|\partial x^{r'} / \partial x^r\|}{\partial x^{i'}}.$$

With each vector (ξ^i) of the tangent centro-affine space $\{A^n\}$ let us associate the object

$$u^i = \frac{\xi^i}{-a_l \xi^l + 1} \tag{1}$$

from the local centro-projective space $\{P^n\}$ (^{1}), and with each covector (ξ_i) from the space $\{B^n\}$, dual to $\{A^n\}$, the object

$$u_i = \xi_i + a_i \tag{2}$$

from the space $\{Q^n\}$, dual to $\{P^n\}$.

The correspondences thus defined are one-to-one, since formulas (1) and (2) can be rewritten in the form

$$\xi^i = \frac{u^i}{a^l_{lu} + 1}, \tag{3}$$

$$\xi_i = u_i - a_i \tag{4}$$

and do not depend on the choice of coordinate system on $\{V^n\}$, for the transformations of the components of a vector $\xi^{i'} = \frac{\partial x^{i'}}{\partial x^i}\xi^i$ from $\{A^n\}$ and of a covector $\xi_{i'} = \frac{\partial x^i}{\partial x^{i'}}\xi_i$ from $\{B^n\}$ entail the transformations

$$u^{i'} = \frac{\frac{\partial x^{i'}}{\partial x^i}u^i}{-\frac{1}{(n+1)}\frac{\partial \ln \det \|\partial x^{r'}/\partial x^r\|}{\partial x^j}u^j + 1},$$

$$u_{i'} = \frac{\partial x^i}{\partial x^{i'}}u_i - \frac{1}{(n+1)}\frac{\partial \ln \det \|\partial x^{r'}/\partial x^r\|}{\partial x^{i'}},$$

of the components of the corresponding objects from $\{P^n\}$ and $\{Q^n\}$.

The object (u_0^i) determines in each $\{P^n\}$ an invariant point, and in each $\{Q^n\}$ an invariant hyperplane

$$u_{0i}^i + 1 = 0,$$

and the object (u_i^0) in $\{P^n\}$ determines the invariant hyperplane

$$u_i^0 u^i + 1 = 0,$$

and in $\{Q^n\}$—an invariant point.

To the sum of vectors $(\xi^i + \eta^i)$ and covectors $(\xi_i + \eta_i)$ there correspond, respectively, the objects

$$u^i \oplus v^i = \frac{u^i + v^i + a_j(u^i v^j + u^j v^i)}{-a_k a_l u^k v^l + 1}, \tag{5}$$

$$u_i \oplus v_i = u_i + v_i - a_i, \tag{6}$$

and to the product of a vector by a number $(\lambda \cdot u^i)$ and of a covector by a number $(\lambda \cdot u_i)$ the objects

$$\lambda \odot u^i = \frac{\lambda u^i}{+a_j(1-\lambda)u^j + 1}, \tag{7}$$

$$\lambda \odot u_i = \lambda u_i + a_i(1-\lambda). \tag{8}$$

2. Let an affine connection be given on $\{V^n\}$, $(\Gamma^p_{jk} = \Gamma^p_{kj})$; then, substituting into the system

$$\frac{\partial \xi^i}{\partial x^k} + \xi^l \Gamma^i_{lk} = 0,$$

which defines parallel displacement of the vector (ξ^i), the value of the latter from (3), we obtain

$$\frac{\partial\left(\dfrac{u^i}{1 + a^p_{pu}}\right)}{\partial x^k} + \frac{u^l}{(1 + a^p_{pu})}\Gamma^i_{jk} = 0$$

or

$$\left(\delta^i_q - \frac{a_q u^i}{(1 + a^p_{pu})}\right)\frac{\partial u^q}{\partial x^k} = \left(\frac{u^i}{1 + a^p_{pu}}\frac{\partial a_j}{\partial x^k} - \Gamma^i_{jk}\right)u^j. \tag{9}$$

The determinant of the obtained system is

$$\det\left\|\delta^i_q - \frac{a_q u^i}{(1 + a^p_{pu})}\right\| = \frac{1}{1 + a^p_{pu}} \neq 0.$$

Solving (9) with respect to the derivatives $\partial u^q / \partial x^k$, we shall have

$$\frac{\partial u^q}{\partial x^k} = -u^j \Gamma^q_{jk} + u^i u^q \left(\frac{\partial a_j}{\partial x^k} - a_i \Gamma^i_{jk}\right). \tag{10}$$

This system defines projective displacements of the local centro-projective spaces $\{P^n\}$ along a curve on the manifold $\{V^n\}$.

Indeed, for every smooth curve

$$x^i = x^i(t) \tag{11}$$

the system (10) gives a system of ordinary differential equations

$$\frac{du^q}{dt} = -u^j \Gamma^q_{jk}\frac{dx^k}{dt} + u^i u^q \left(\frac{\partial a_j}{\partial x^k} - a_i \Gamma^i_{jk}\right)\frac{dx^k}{dt},$$

which, under the initial conditions $t = t_0,\ \ u^q|_{u^q = u^q_0}$, has a unique solution.

It should be noted that the projective displacement of $\{P^n\}$ from the point M_0 to the point M_1 along the curve (11) can be carried out by successive application of the following three transformations: 1) transformation (3) at the point M_0; 2) transformation of parallel translation in the given

of the affine connection from the point M_0 to the point M_1 along the curve (11); 3) transformation (1) at the point M_1.

In particular, if the curve (11) is regarded as closed ($M_0 \equiv M_1$), then from the remark made above we obtain:

The holonomy group of the projective transports defined by the system (10) is a group similar to the corresponding holonomy group of the affine-connection space under consideration.

Next, considering the system $\frac{\partial \xi_i}{\partial x^k} - \xi_l \Gamma^l_{ik} = 0$ and substituting in place of (ξ_i) its value from (4), we obtain

$$\frac{\partial u_i}{\partial x^k} = u_l \Gamma^l_{ik} + \left(\frac{\partial a_i}{\partial x^k} - a_l \Gamma^l_{ik} \right).$$

Analogously to the preceding case, this system defines a transport of the spaces $\{Q^n\}$ along the curve (11), and here again the holonomy group of these transports is similar to the holonomy group of the affine-connection space.

The operations (5) and (7), (6) and (8) are commutative with the operations of transport along a curve.

3. If the connection Γ^p_{ik} is Riemannian and g_{ij} is the fundamental metric tensor, then in each $\{P^n\}$ there arises the invariant hyperquadric

$$[a_i a_j - g_{ij}] u^i u^j + 2 a_i u^i + 1 = 0,$$

and the invariant hyperplane

$$a_i u^i + 1 = 0,$$

defined by the object (a_i), becomes the polar hyperplane of the central point $(u^i = 0)$.

The system (10) now defines projective-metric transports. Moreover, all projective-metric transports of $\{P^n\}$ that leave the point $(u^i = 0)$ invariant can always be defined by this system, since no restrictions are imposed on the components of the tensor (g_{ij}) and of the object (a_i).

Let us also note that if one takes

$$a_i = -\frac{1}{(n+1)} \Gamma^a_{ai},$$

then the system (10) assumes the form

$$\frac{\partial u^q}{\partial x^k} = -u^j \Gamma^q_{jk} - \frac{1}{(n+1)} u^i u^q \left(\frac{\partial \Gamma^a_{aj}}{\partial x^k} - \Gamma^a_{ai} \Gamma^i_{jk} \right), \tag{12}$$

and the transport of the local centro-projective spaces $\{P^n\}$ will be invariantly determined by the original affine-connection space. In addition, the object (Γ^a_{ai}) will determine in each $\{P^n\}$ the invariant hyperplane

$$-\frac{1}{(n+1)} \Gamma^a_{aj} u^j + 1 = 0$$

and thus will acquire a concrete geometric meaning.

4. Suppose now that the Ricci tensor (R_{ij}) of the affine-connection space under consideration has a nondegenerate symmetric part

$$\sigma_{ij} = \frac{R_{ij} + R_{ji}}{2}; \tag{13}$$

then in each $\{P^n\}$ an invariant hyperquadric is determined

$$\left[\frac{1}{(n+1)^2} \Gamma^a_{ai} \Gamma^b_{bj} - \frac{\sigma_{ij}}{n-1} \right] u^i u^j - \frac{2}{(n+1)} \Gamma^a_{ai} u^i + 1 = 0,$$

but, this time, the projective-metric transfers will be given not by system (12), but by the system

$$\frac{\partial u^q}{\partial x^k} = -u^i \tilde{\Gamma}^q_{ik} - \frac{1}{(n+1)} u^i u^q \left(\frac{\partial \Gamma^a_{aj}}{\partial x^k} - \Gamma^a_{ai} \tilde{\Gamma}^i_{jk} \right), \tag{14}$$

where $\tilde{\Gamma}^q_{jk}$ is the Riemannian connection constructed for the tensor (13).

It is not difficult to show that formulas (12) and (14) coincide if and only if the original space is a space of constant curvature.

5. In the latter case one can always pass to such a coordinate system in which

$$\Gamma^q_{jk} = \tilde{\Gamma}^q_{jk} = -\frac{\delta^q_j (c_{ik} x^i + c_k) + \delta^q_k (c_{ij} x^i + c_j)}{c_{ij} x^i x^j + 2 c_i x^i + 1}$$

$$(c_{il}, c_i, c = \text{const}) \quad \text{and} \quad \det \| c_i c_j - c_{ij} \| \neq 0.$$

This coordinate system is determined up to arbitrary fractional-linear transformations (²), and equations (12) in these coordinates take the form

$$\frac{\partial u^q}{\partial x^k} = \frac{u^i\left[\delta_j^q(c_{ik}x^i + c_k) + \delta_k^q(c_{ij}x^i + c_j) + c_{jk}u^q\right]}{c_{ij}x^ix^j + 2c_ix^i + 1}. \tag{15}$$

Similarly, for the object (u_j) we obtain

$$\frac{\partial u_j}{\partial x^k} = \frac{-u_j(c_{kl}x^l + c_k) - u_k(c_{jl}x^l + c) + c_{jk}}{c_{il}x^ix^l + 2c_ix^i + 1}. \tag{16}$$

In conclusion we note that, if one does not assume $\det\|c_ic_j - c_{ij}\| \neq 0$, then formulas (15) and (16) determine the transfer of the objects (u^i) and (u_i) for symmetric projective-Euclidean spaces (^{3}).

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