MATHEMATICS

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1960

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Abstract

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MATHEMATICS

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ON THE COHOMOLOGY OF COMPLEX HOMOGENEOUS MANIFOLDS

(Presented by Academician P. S. Aleksandrov, 30 XII 1959)

Let X be a homogeneous simply connected compact complex manifold. Denote by $H^q(X,\Omega^p)$, or $H^{p,q}(X)$, the cohomology group of the space X with coefficients in the sheaf Ω^p of germs of holomorphic p-forms on X, and set $H''(X)=\sum_{p,q} H^{p,q}(X)$. In the case when X is a Kähler manifold, we have ([1])*

$$H^q(X,\Omega^p)=0 \quad \text{for } p\neq q; \tag{1}$$

$$H^p(X,\Omega^p)=H^{2p}(X,C).$$

For the general case, Bott' s formula ([2]) is known, reducing the group $H^{p,q}(X)$ to the cohomology groups of certain Lie algebras.

To each non-Kähler homogeneous manifold one can associate in a unique way a certain Kähler manifold Y. In this case there exists a fibration $X\to Y$, whose fiber is the torus T^{2n} of dimension $2n$. In the present paper it is proved that $H''(X)$ is isomorphic, as a bigraded space, to the cohomology group of the algebra $H^*(Y,C)\otimes H^*(T^{2n},C)$, endowed with a certain differential and a certain bigrading. On a manifold X of the indicated type a semisimple complex group G acts transitively, and its maximal compact subgroup M, so that $X=G/U=M/V;\ V=M\cap U$. Denote by $\mathfrak{g},\mathfrak{u},\mathfrak{v}$ the Lie algebras of the groups G,U,V, and by $\mathfrak{v}^c$ the complex envelope of the algebra $\mathfrak{v}$. Let S' be some subset of the system S of simple roots of the algebra $\mathfrak{g}$, and let $[S']$ be the set of positive roots generated by it. Denote by $\mathfrak{v}(S')$ the subalgebra in $\mathfrak{g}$ generated by all such root vectors e_α that α or $-\alpha\in[S']$, and by $\mathfrak{h}(S')$ the orthogonal complement to the subspace spanned by $[S']$ in the Cartan subalgebra, and by $\mathfrak{n}(S')$ the subalgebra spanned by those e_α for which $\alpha>0$ and $\alpha\notin[S']$. Wang showed ([3]) that for every manifold X of the type under consideration there exist an $S'\subseteq S$, a decomposition $\mathfrak{h}(S')=\mathfrak{h}_v+\mathfrak{h}_c$, where $\mathfrak{h}_v$ and $\mathfrak{h}_c$ are rational subspaces, and a complex subspace $\mathfrak{w}\subset\mathfrak{h}_c$, $\mathfrak{w}+\overline{\mathfrak{w}}=\mathfrak{h}_c$, such that

$$\mathfrak{u}=\mathfrak{v}(S')+\mathfrak{h}_v+\mathfrak{w}+\mathfrak{n}(S'),$$

$$\mathfrak{v}^c = \mathfrak{v}(S') + \mathfrak{h}_v.$$

Put

$$[\mathfrak{u}] = \mathfrak{u} + \overline{\mathfrak{w}}.$$

* By $H^k(X, C)$ we denote the cohomology groups of the space X with coefficients in the field of complex numbers C.

and denote by $[U]$ the connected complex subgroup of the group G corresponding to this subalgebra. Then it follows from Wang' s results ([3]) that $Y = G/[U]$ is a simply connected compact Kähler manifold. Denote by $[\mathfrak{v}]$ the Lie algebra of the group $[V] = [U] \cap M$. We have

$$[\mathfrak{v}]^c = \mathfrak{v}^c + \mathfrak{h}_c.$$

Let us note that, for different choices of $\mathfrak{w} \subset \mathfrak{h}_c$, different complex structures may be induced on X by the fibration (G, X, U).

We formulate the main result.

The space $\overline{\mathfrak{w}}$ is contained in the center of the algebra $[\mathfrak{v}]^c$. Therefore the conjugate space $\overline{\mathfrak{w}}^*$ is identified with a subspace of the space $H^1([V], C)$. Transgression in the principal fiber space $(M, Y, [V])$ induces a linear map $\tau : \overline{\mathfrak{w}}^* \to H^2(Y, C)$. Consider the algebra

$$\theta = \bigwedge \overline{\mathfrak{w}}^* \otimes H(Y, C) \otimes \bigwedge \mathfrak{w}^*$$

with the double grading such that the elements of $\bigwedge^l \overline{\mathfrak{w}}^* \otimes 1 \otimes 1$ have bidegree $(l, 0)$, the elements of $1 \otimes H^{2p}(Y, C) \otimes 1$ have bidegree (p, p), and the elements of $1 \otimes 1 \otimes \bigwedge^m \mathfrak{w}^*$ have bidegree $(0, m)$, and define in it a differential δ for which

$$\delta \mathfrak{w}^* = \delta H(Y) = 0, \qquad \delta = \tau \text{ on } \overline{\mathfrak{w}}^*.$$

Theorem. The group $H''(X)$, as a bigraded space, is isomorphic to $H(\theta) = \bigwedge \mathfrak{w} \otimes H(H(Y, C) \otimes \bigwedge \overline{\mathfrak{w}}^*)$.

In proving the theorem we start from Borel' s formula ([2])

$$H^q(X, \Omega^p) = H^q(\mathfrak{u}, \mathfrak{v}^c, \bigwedge^p (\mathfrak{g}/\mathfrak{u})^*),$$

where on the right stands the cohomology group of the Lie algebra $\mathfrak{u}$ relative to the subalgebra $\mathfrak{v}^c$ with coefficients in the representation of the algebra $\mathfrak{u}$ in the space $\bigwedge(\mathfrak{g}/\mathfrak{u})^*$, induced by the adjoint representation. Identify the spaces $\mathfrak{g}/\mathfrak{u}$ and $\mathfrak{g}/[\mathfrak{u}]$ with the complements $\overline{\mathfrak{n}}_* = \mathfrak{n}(S') + \overline{\mathfrak{w}}$ and $\overline{\mathfrak{n}}(S')$ to $\mathfrak{u}$ and $[\mathfrak{u}]$ in the algebra $\mathfrak{g}$.

Then we have

$$\bigwedge(\overline{\mathfrak{n}}_*)^* = \bigwedge(\mathfrak{n}(S'))^* \otimes \bigwedge \overline{\mathfrak{w}}^*,$$

and the operators from $[\mathfrak{u}]$ act on $\bigwedge \overline{\mathfrak{w}}^*$ trivially, while on $\bigwedge \overline{\mathfrak{n}}(S')^*$ their action is induced by the adjoint representation of $\mathfrak{u}$ in $\mathfrak{g}$.

We shall denote by $C(L, L', P)$ the algebra of relative cochains of the Lie algebra L with respect to the subalgebra L', with coefficients in the representation P, and by d the differential in this algebra.

Let

$$A = C(\mathfrak{u}, \mathfrak{v}^c, \bigwedge(\overline{\mathfrak{n}}_*)^*) \quad \text{and} \quad B = C(\mathfrak{u}, \mathfrak{v}^c, \bigwedge \overline{\mathfrak{n}}(S')^*).$$

We have

$$A = B \otimes \bigwedge \overline{\mathfrak{w}}^*.$$

Let $x \in \overline{\mathfrak{w}}^* \subset A$. Compute dx. For this we choose in $\mathfrak{g}$ a basis of root vectors and roots. Then, by the definition of the differential in A ([4]), the element $[dx(e_\alpha)](e_{-\beta})$, where $e_\alpha \in \mathfrak{n}(S')$, $e_{-\beta} \in \overline{\mathfrak{n}}(S')$, is equal to 0 if $\alpha \neq \beta$, and is equal to $x(\alpha)$ if $\alpha = \beta$. It is obvious that $[dx](\bar{h}) = 0$, $\bar{h} \in \overline{\mathfrak{w}}$, and $dx(h) = 0$, $h \in \mathfrak{w}$. Compute $H(B)$. As follows from ([2]), the graded algebra associated with $H(B)$ is isomorphic to the term ${}'E_\infty$ of a certain spectral sequence in which

$${}'E_2^q = \sum H^{p,q}(Y, C) \otimes \bigwedge \mathfrak{w}^*.$$

From dimension considerations and by virtue of (1), ${}'E_2 = {}'E_\infty$, and the algebra E_∞ is isomorphic to the algebra $H(B)$. With the aid of the usual filtration of the algebra A by degree in the algebra B, we construct the cohomological spectral sequence (E_r, d_r).

In this case

$$E_2 = \Lambda\overline{\mathfrak{w}}^* \otimes H(B) = \Lambda\overline{\mathfrak{w}}^* \otimes H(Y, C) \otimes \Lambda\mathfrak{w}^*,$$

and the element $d_r x$, where $x \in \mathfrak{w}^*$, belongs to $H(B)$ and is equal to the cohomology class of the cycle dx. It can be proved that in the given spectral sequence the differentials d_r $(r > 2)$ are trivial. Thus,

$$H(A) = H(\Lambda\overline{\mathfrak{w}}^* \otimes H(Y, C)) \otimes \Lambda\mathfrak{w}^*,$$

where the differential δ, with respect to which the homology group is taken, is induced by the differential d computed above. It remains to compute δ.

Put

$$D = C([\mathfrak{u}], [\mathfrak{v}]^c, \Lambda\overline{\mathfrak{n}}(S'_\gamma)^*).$$

As was shown, $d\overline{\mathfrak{w}}^* \subset D^{1,1}$. Define a mapping

$$P : D^{p,q} \to C^{p+q}(\mathfrak{g}, [\mathfrak{v}]^c, c)$$

as follows:

$$Pf(x_1, \dots, x_{p+q}) = \begin{cases} 0, & \text{if } x_p \in \overline{[\mathfrak{u}]}, \\ [f(x_1, \dots, x_p)](\hat{x}_{p+1}, \dots, \hat{x}_{p+q}), & \text{if } x_p \in [\mathfrak{u}], \end{cases}$$

where the x_i are arranged so that if $x_i \in \overline{[\mathfrak{u}]}$, then also $x_j \in \overline{[\mathfrak{u}]}$ for $j > i$, and $\hat{x}_k$ is equal to the coset of the element x_k in the space $\mathfrak{g}/[\mathfrak{u}]$.

Consider the diagram

$$\begin{array}{ccc} D^{p,q}(Y) & \xrightarrow{p} & C^{p+q}(Y) \\ \sigma'' \downarrow & & \downarrow \sigma'' \\ I^{p,q}(Y) & \xrightarrow{i} & I^{p+q}(Y) \end{array}$$

where $I^{p,q}(Y)$ are invariant forms on Y of bidegree (p, q) (degree r), σ is the known identification ([4]), and i is the obvious inclusion. One can check preservation of bidegree under the mapping σ; consequently, one can define

$$\sigma'' = i^{-1}\sigma P.$$

It is easy to verify, by directly comparing the corresponding formulas, that

$$\sigma'' d = d''\sigma'',$$

where d is the differential in the algebra A, and d'' is differentiation with respect to $\overline{z}$ in $I(Y)$.*

Since the manifold Y is Kähler, it is known ([5]) that, upon passing to cohomology algebras, the mapping i in $I^{p,q}(Y)$ and in $I^{p+q}(Y)$ induces an isomorphism of the algebras $H''(Y)$ and $H(Y, C)$. Therefore the mapping P also induces an isomorphism P^* of the algebras $H''(Y)$ and $H(Y, C)$, computed by means of the complexes D and $C(\mathfrak{g}, [\mathfrak{v}], C)$.

As was noted above, $\overline{\mathfrak{w}}^*$ is identified with the subspaces $j\overline{\mathfrak{w}}^*$ and $j^*\mathfrak{w}^*$ of the spaces $C^1([\mathfrak{v}]^c)$ and $H^1([\mathfrak{v}]^c)$. Let $x \in \overline{\mathfrak{w}}^*$; then $jx \in C^1([\mathfrak{v}]^c) \subset C(\mathfrak{g})$, and let $\tilde{d}$ be the differential in the latter group. Then $(\tilde{d}x)(e_\alpha, e_\beta)$ is equal to zero if $\alpha + \beta \neq 0$, and is equal to $x(\alpha)$ if $\alpha + \beta = 0$. Comparing this formula with the value computed above for dx in the algebra A, we see that the diagram

$$\begin{array}{ccc} \Lambda^1\overline{\mathfrak{w}}^* & \xrightarrow{j} & C^1([\mathfrak{v}]^c) \\ d \downarrow & & \downarrow \tilde{d} \\ D^{1,1} & \xrightarrow{p} & C^2(\mathfrak{g}, [\mathfrak{v}]^c, C) \end{array}$$

is commutative. Upon passing in the lower row to cohomology groups, we obtain the diagram

$$\begin{array}{ccc} \Lambda^1\overline{\mathfrak{w}}^* & \xrightarrow{j^*} & H^1([\mathfrak{v}], C) \\ d_2 \downarrow & & \downarrow \tau \\ H^{1,1}(Y) & \xrightarrow{P^*} & H^2(Y, C), \end{array}$$

* This construction is also applicable to the original manifold X;

it follows directly from it that

$$H(\mathfrak{u}, \mathfrak{v}^c, \Lambda(\mathfrak{g}/\mathfrak{u})^*)$$

is isomorphic to the algebra of d''-cohomologies of the algebra of invariant forms on the manifold X. Therefore Bott' s formula is equivalent to the assertion that $H''(X)$ can be computed by means of invariant forms on X.

where P^* is an isomorphism, d_2 is the second differential in the spectral sequence (E_r, d_r) defined above, and τ is the transgression in the fiber space $(M, Y, [V])$. We map the algebra $E_2\, \Lambda\mathfrak{m}^* \otimes H''(Y) \otimes \Lambda\mathfrak{p}^*$ into the algebra $\theta = \Lambda\mathfrak{m}^* \otimes H(Y, C) \otimes \Lambda\mathfrak{p}^*$ by means of $1 \otimes P^* \otimes 1$. This isomorphism carries d_2 into the differential τ of the algebra θ and induces an isomorphism of the groups $H(A)$ and $H(\theta)$. The theorem is proved.

Corollary. $H^q(X, \mathfrak{D})$—the cohomology group of the space X with coefficients in the sheaf of germs of analytic functions—is isomorphic to $\Lambda^q\mathfrak{m}^*$.

As an example, consider the space $SU(3)$, endowed with a left-invariant complex structure. Let

$$P(X, s, t) = \sum \dim H^{p,q}(X) s^q t^p.$$

Depending on the choice of complex structure on $SU(3)$, the polynomial $P(SU(3), s, t)$ will be equal either to $(1 + st)(1 + s^2t^3)(1 + s)$ or to $(1 + st^2)(1 + st + s^2t^2)(1 + s)$. This example was analyzed by Bott ([2]), but the values he obtained for $\dim H^{p,q}(SU(3))$ do not coincide with those computed above.

In conclusion, the author expresses his gratitude to A. L. Onishchik, who supervised this work.

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Received
25 XII 1959

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Note: Figure translations are in progress. See original paper for figures.