THE MULTIPLICATIVE GROUP OF A DIVISION RING

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Abstract

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THE MULTIPLICATIVE GROUP OF A DIVISION RING

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Let K be an associative noncommutative division ring, and Z the center of K. The set of all nonzero elements of the division ring K forms a group under multiplication, which we shall denote by K^*. If Z^* is the set of all nonzero elements of Z, then Z^* is the center of the group K^*. The group K^* has so far been little studied. Hua ([1,2]) proved that K^* is not a (finite) solvable group and that the factor group K^*/Z^* has no center. Scott ([3]) generalized this, proving that the factor group K^*/Z^* has no nontrivial abelian normal divisors.

In the present article these results are generalized. The following main theorem holds:

Theorem 1. *The group K^* is not locally nilpotent. Moreover, every locally nilpotent normal divisor of the group K^* is contained in the center Z^*. Consequently, the factor group K^*/Z^* has no nontrivial locally nilpotent normal divisors.*

In other words, using the terminology and results from the works of B. I. Plotkin ([4,5]), Z^* is the upper radical of the group K^*. The group K^* is not a radical group. In particular, K^* is not an RN^*-solvable group, i.e., K^* does not possess an ascending solvable normal series. Indeed, every RN^*-solvable normal divisor of the group K^* is contained in the center Z^*.

Thus, one may say that the division ring K is strongly noncommutative.

The proof of the theorem is based on the following four lemmas. Denote $[a,b] = aba^{-1}b^{-1}$ for nonzero elements a and b of the division ring K.

Lemma 1. *Let x and y be elements of the division ring K such that*

$$[y,x] = yxy^{-1}x^{-1} = \lambda \neq 1,$$

where λ commutes with each of the elements x and y. Let Λ be the ring of polynomials in λ with integral coefficients (the integers being taken from the prime field P of the division ring K). Then, if x is algebraic over Λ, λ is a root of unity. If, in addition, y is also algebraic over Λ, then the characteristic of the division ring K is zero.

Proof. Let r be the least degree of an equation over Λ satisfied by the element x:

$$f(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_{rx}^r = 0, \qquad \alpha_i \in \Lambda.$$

Using the relation $yx = \lambda xy$, we obtain

$$yf(x)y^{-1} = \alpha_0 + \alpha_1 \lambda x + \dots + \alpha_r \lambda^r x^r = 0.$$

Therefore,

$$\alpha_1(\lambda - 1) + \alpha_2(\lambda^2 - 1)x + \dots + \alpha_r(\lambda^r - 1)x^{r-1} = 0.$$

If $\lambda^r \neq 1$, then x satisfies an equation of degree less than r, which is impossible. Suppose now that the element y is also algebraic over Λ and that the characteristic of the skew field K is finite. Since λ is a root of unity, Λ is a finite field. Further, the set T of all finite sums of the form $\sum a_{ij} x^i y^j$, $a_{ij} \in \Lambda$, is a finite skew field and, consequently, a field. Since x and $y \in T$, we have $yx = xy$, which contradicts the original assumption $\lambda \neq 1$.

Lemma 2. Let x, y, λ, Λ be defined as in Lemma 1. Define

$$x_1 = [y, 1 + x], \qquad x_{i+1} = [y, x_i], \qquad i = 1, 2, 3, \dots$$

If $x_n = 1$ for some n, then x is algebraic over Λ. Further, let m be an integer from the prime field P of the skew field K. Define analogously for the element mx

$$(mx)_1 = [y, 1 + mx], \qquad (mx)_{i+1} = [y, (mx)_i], \qquad i = 1, 2, 3, \dots$$

If for every integer m from P there exists an index N_m such that $(mx)_{N_m} = 1$, then the characteristic of the skew field K is finite.

Proof. We first note that, by virtue of the relation $yx = \lambda xy$ and the commutativity of the element λ with x and y, we obtain $yf(x) = f(\lambda x)y$ for an arbitrary rational function $f(x)$ with integer coefficients. If $f(x) \neq 0$, then $[y, f(x)] = f(\lambda x)(f(x))^{-1}$. Hence

$$x_1 = [y, 1+x] = (1+\lambda x)(1+x)^{-1}, \qquad x_2 = [y, x_1] = (1+\lambda^2 x)(1+\lambda x)^{-2}(1+x);$$

in general,

$$x_n = [y, x_{n-1}] = \prod_{i=0}^{n} (1 + \lambda^i x)^{\left[n \atop i \right] \cdot (-1)^{n-i}},$$

where $\left[{n \atop i} \right] = \dfrac{n!}{i!(n-i)!}$.

If now $x_n = 1$, then the equality

$$(\lambda - 1)^n + a_1 x + a_2 x^2 + \dots + a_r x^r = 0, \qquad a_i \in \Lambda, \qquad r = 2^{n-1} - 2,$$

holds, i.e. x is algebraic over Λ.

Let for every integer m, $(mx)_{N_m} = 1$. Then mx satisfies a polynomial over Λ, namely

$$L_m(t) = (\lambda - 1)^{N_m} + a_1^{(m)} t + a_2^{(m)} t^2 + \dots + a_{r_m}^{(m)} t^{r_m},$$

where

$$a_i^{(m)} \in \Lambda, \qquad r_m = 2^{N_m - 1} - 2.$$

Let q be the least degree of a polynomial over Λ satisfied by the element x; $G(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_{qx}^q = 0$, $b_i \in \Lambda$. Then q is the least degree of a polynomial over Λ satisfied by the element mx; as such a polynomial one may take

$$G_m(t) = b_0 m^q + b_1 m^{q-1} t + b_2 m^{q-2} t^2 + \dots + b_{qt}^q.$$

Applying the division theorem (see ([6]), p. 30) to the polynomials $L_m(t)$ and $G_m(t)$, we find a polynomial $Q_m(t)$ with coefficients in Λ such that

$$b_q^{k_m} L_m(t) = Q_m(t) G_m(t), \qquad \text{where } k_m = r_m - q + 1.$$

Comparing constant terms, we obtain $b_q^{k_m} (\lambda - 1)^{N_m} = c_0^{(m)} b_0 m^q$, where $c_0^{(m)}$ is the constant term in $Q_m(t)$. Consequently, $b_q^{k_m} (\lambda - 1)^{N_m}$ is divisible by m.

Suppose now that the characteristic of the field K is zero. Let λ be a primitive l-th root of unity; $1 = s_1, s_2, \dots, s_h$ are natural numbers less than l and relatively prime to l. For each polynomial $f(\lambda)$ in λ with integral coefficients, define the norm

$$\operatorname{Norm} f(\lambda) = \prod_{j=1}^{h} f(\lambda^{s_j}).$$

It is clear that the norm has the multiplicative property and that its values are integers.

Since $(\mathrm{Norm}\, b_q) \cdot (\mathrm{Norm}(\lambda - 1))$ has only a finite number of prime divisors, there is a prime number m which does not divide $(\mathrm{Norm}\, b_q) \cdot (\mathrm{Norm}(\lambda - 1))$. Then m does not divide $b_q^{k_m}(\lambda - 1)^{N_m}$. This is a contradiction.

The following result is obvious, but since it is essential, we formulate it as a lemma.

Lemma 3. *In every nonabelian nilpotent group one can find elements x, y such that $[y, x] = yxy^{-1}x^{-1} = \lambda \neq 1$, where λ commutes with the elements x and y.*

Lemma 4. *Every abelian normal divisor of the group K^* is contained in the center Z^*.*

This follows from the Cartan–Brauer–Hua theorem (see (7), p. 186).

Proof of Theorem 1. Let H be a locally nilpotent normal divisor and $H \nsubseteq Z^*$. Then, by Lemmas 3 and 4, in H there are elements a, b such that $[a, b] = aba^{-1}b^{-1} = \lambda \neq 1$, where λ commutes with the elements a and b. Let m be an arbitrary integer from the prime field P of the field K. Define $(mb)_1 = [a, 1+mb]$, $(mb)_{i+1} = [a, (mb)_i]$, $i = 1, 2, 3, \ldots$. It is clear that $(mb)_1 \in H$, and, by the local nilpotence of the group H, the elements a and $(mb)_1$ generate a nilpotent subgroup.

Consequently, $(mb)_{N_m} = 1$ for some N_m. Now it follows from Lemma 2 that the element b is algebraic over Λ and the characteristic of the field K is finite. Similarly it is proved that the element a is also algebraic over Λ. But then Lemma 1 implies that the characteristic of the field K is zero. This is a contradiction.

Remark. In the proof of the theorem, the local nilpotence of the group H is not quite essential, since only the fact is used that any two elements of the group H generate a nilpotent subgroup. Groups in which any two elements generate a nilpotent subgroup were called weakly nilpotent by B. I. Plotkin (5). However, it is still unknown whether there exist weakly nilpotent but not locally nilpotent groups.

The center Z^* of the group K^* has yet another interesting property, namely, it is a primary normal divisor in the sense of K. K. Shchukin (8). Recall that a normal divisor N of a group G, different from the group itself, is called a **primary normal divisor** if from the relation $[A, B] \subseteq N$, where $[A, B]$ is the mutual commutator of the invariant subgroups A and B in G, it follows that at least one of the subgroups A, B is contained in N. A group G is called **primary** if its identity subgroup is a primary normal divisor in G. In (8) it is proved that N is a primary normal divisor of the group G if and only if the factor group G/N is primary.

Theorem 2. *Z^* is a primary normal divisor of the group K^* and, consequently, K^*/Z^* is a primary group.*

First note that, by the Cartan–Brauer–Hua theorem (7), the following lemma holds:

Lemma 5. *If N is a normal divisor of the group K^* and $N \not\subseteq Z^*$, then the centralizer of the subgroup N in K^* coincides with Z^*.*

Proof of Theorem 2. Let A and B be normal subgroups of the group K^*, with $B \not\subseteq Z^*$. Then from $[A, B] \subseteq Z^*$ it follows at once that $[[A, A], B] = 1$, i.e., $[A, A]$ is contained in the centralizer of the subgroup B in K^*. By Lemma 5, $[A, A] \subseteq Z^*$. In view of Theorem 1, $A \subseteq Z^*$.

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Note: Figure translations are in progress. See original paper for figures.