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Abstract

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THE STRUCTURE OF RINGS OF CONTINUOUS FUNCTIONS ON A CIRCLE WITH TWO GENERATORS

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Consider on the circle $|\zeta| = 1$ the ring $[\varphi, f]$ of continuous functions with two generators $\varphi(\zeta)$ and $f(\zeta)$, closed with respect to uniform convergence. Suppose that φ and f separate the points of the circle, i.e., for every pair of points $\zeta_1 \neq \zeta_2$, $|\zeta_i| = 1$ $(i = 1, 2)$, either $\varphi(\zeta_1) \neq \varphi(\zeta_2)$ or $f(\zeta_1) \neq f(\zeta_2)$. We shall determine what conditions the generators of the ring φ and f must satisfy in order that the ring $[\varphi, f]$ not coincide with the ring C of all continuous functions. Wermer, in papers ([1,2]), established what the generator f must be in order that $[\varphi, f] \neq C$ when $\varphi(\zeta)$ has a special form: $\varphi(\zeta) = \zeta$, or $\varphi(\zeta) = \zeta^2$. An analogous result was obtained by us in the note ([3]), when $\varphi(\zeta) = \zeta^n$, n being any natural number. In that note we studied a system of equations which is also used in the present work. A similar system is used by Wermer in ([4,5]).

In papers ([4,5]) Wermer, abandoning the special form of one of the generators, imposes rather strong conditions already on both generators. Namely, he requires that:

A. Both functions $\varphi(\zeta)$ and $f(\zeta)$ be analytic on the circle $|\zeta| = 1$ and $\varphi'(\zeta) \neq 0$, $|\zeta| = 1$.

Under this assumption Wermer, without loss of generality, could regard the following condition as satisfied:

B. On the circle there is only a finite set M of points such that for every $\zeta_1 \in M$ there is another point $\zeta_2 \in M$ for which $\varphi(\zeta_1) = \varphi(\zeta_2)$.

In the present work we abandon Wermer' s restriction A, replacing it by the condition:

A$'$. The function $\varphi(\zeta)$ has a derivative satisfying a Hölder condition, and $f(\zeta)$ satisfies a Hölder condition.

In order not to make the exposition cumbersome, we shall everywhere assume that condition B is also satisfied; however, it can be replaced by a more general one. For example, we may allow the point $\varphi(\zeta)$ to run a finite number of times along an arc of the curve $\gamma : \lambda = \varphi(\zeta)$, when the point ζ describes the circle $|\zeta| = 1$ once, although the support of the curve still has a finite number of self-intersection points. Exactly the same arguments apply to certain kinds of curves whose supports have an infinite number of self-intersection points and under more general smoothness assumptions.

Theorem 1. *If $\varphi(\zeta)$ and $f(\zeta)$ satisfy conditions A' and B, then $[\varphi, f] \neq C$ if and only if the curve in the space of two complex variables $\Gamma : z_1 = \varphi(\zeta),\ z_2 = f(\zeta),\ |\zeta| = 1$, bounds a piece of an analytic surface.*

Theorem 2. Let $\varphi(\zeta)$ and $f(\zeta)$ satisfy conditions A' and B. Then $[\varphi, f] \neq C$ if and only if

$$\int_{|\zeta|=1} \varphi^m(\zeta) f^n(\zeta) \varphi'(\zeta)\, d\zeta = 0, \qquad n, m \geqslant 0.$$

Theorems 1 and 2 are a generalization of Wermer' s theorems, in which he assumes that φ and f satisfy condition A^*.

In the proof, an important role is played by the integral representation we found for functions connected with the solution of the above-mentioned system of equations (Theorem 4). This representation makes it possible to establish Theorem 3, which, from the form of the curve γ, in a number of cases permits one to assert that $[\varphi, f] = C$ for any f.

Theorem 3. Let $\varphi(\zeta)$ satisfy condition B and have a continuous derivative, and let the component D_0 of the complement of γ that contains ∞ be such that the complement of $\overline{D}_0$ is disconnected. Then $[\varphi, f] = C$ for any $f(\zeta)$ satisfying Helder' s condition.

We outline the course of the proof. Let $[\varphi, f] \neq C$. Then there exists a nontrivial functional on C that vanishes on $[\varphi, f]$, i.e., there is a measure $d\mu(\zeta) \neq 0$ for which

$$\int_{|\zeta|=1} f^n(\zeta) \varphi^m(\zeta)\, d\mu(\zeta) = 0, \qquad n, m \geqslant 0.$$

The function $\lambda = \varphi(\zeta)$ maps the circle onto the curve γ. Then $f(\zeta) = f(\varphi^{-1}(\lambda)) = \tilde{f}(\lambda)$ is a function of λ, single-valued and continuous and satisfying Helder' s condition everywhere except at the points $\widetilde{M}$ of self-intersection of γ, where $\tilde{f}(\lambda)$ is multivalued. But if α is a simple arc of γ with endpoints belonging to the set $\widetilde{M}$, then $\tilde{f}(\lambda)$ is continuous and satisfies Helder' s condition on the closed arc α. Denote the measure

$$\prod_{\zeta_j \in M} (\varphi(\zeta) - \varphi(\zeta_j))\, d\mu(\zeta)$$

by $d\mu_1(\zeta)$; evidently,

$$\int_{|\zeta|=1} f^n(\zeta)\varphi^m(\zeta)\, d\mu_1(\zeta) = 0, \quad n, m \geqslant 0,$$

and from the property of separability of the circle by the functions f and φ it is not difficult to derive that $d\mu_1(\zeta) \neq 0$. Put $d\tilde{\mu}(\lambda) = d\mu_1(\varphi^{-1}(\lambda))$. Then

$$\int_\gamma \tilde{f}^n(\lambda)\lambda^m\, d\tilde{\mu}(\lambda) = 0, \qquad m, n \geqslant 0 \tag{1}$$

(in what follows we shall omit the tilde sign $\sim$). Consider the function

$$\Phi^{(n)}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f^n(\lambda)\, d\mu(\lambda)}{\lambda - z}, \qquad n = 0, 1, 2, \ldots$$

It is analytic in the components of the complement of the curve γ and in the component of the complement D_0 that contains ∞; $\Phi^{(n)}(z) \equiv 0$ by virtue of (1). Let G_1 be one of the components of the complement of $\overline{D}_0$. All components of the complement of γ inside G_1 in which $\Phi^{(n)}(z) \equiv 0$, as well as D_0, will be called domains of zero rank; the union of their closures we denote by Ω_0. If a component $D_1 \subset G_1$ of the complement of γ has a common boundary arc α with Ω, then $\Phi_1^{(n)}(\lambda) = \pm\mu'(\lambda) f^n(\lambda)$, where $\Phi_1^{(n)}(\lambda)$ are the boundary values

* **Note added in proof.** We note that in Wermer's proofs the analyticity of the generators is repeatedly and essentially used. M. V. Fedoryuk (*), referring to D. A. Anosov, gives a lemma stating that every closed, twice continuously differentiable curve in C^n projects parallel to a suitable complex $(n-1)$-dimensional hyperplane onto the complex plane with a finite number of exceptional points. Erroneously supposing that analyticity is used by Wermer only in condition B, Fedoryuk formulates Wermer' s theorems for twice continuously differentiable generators.

to α of the function $\Phi^n(z)$ in D_1. Denoting $\pm\mu'(\lambda)$ by $x_1(\lambda)$, and $f^n(\lambda)$ by $y_1^n(\lambda)$, we shall have

$$\Phi_1^{(n)}(z) = x_1(z) y_1^n(z), \qquad z \in D_1,$$

where $x_1(z), y_1(z)$ are functions whose boundary values on α are almost everywhere equal to $x_1(\lambda)$ and $y_1(\lambda)$.

It can be shown analogously that in every component $D \subset G_1$ of the complement to γ

$$\Phi^{(n)}(z) = x_1(z) y_1^n(z) + \cdots + x_k(z) y_k^n(z),$$

and the equalities $y_j(z) = y_i(z)$ occur only for a discrete set of points inside D. We shall say in this case that the domain D has rank k. The aggregate of the closures of the domains of rank k will be denoted by Ω_k. We prove that the function

$$R_k^{(n)}(z) = y_1^n(z) + \cdots + y_k^n(z)$$

is analytic and bounded by the constant kM^n inside Ω_k, where $M = \max_{|\zeta|=1} |f(\zeta)|$. Using this, we find an integral representation for $R_k^{(n)}(z)$

$$R_k^{(n)}(z) = y_1^n(z)+\cdots+y_k^n(z) = \frac{1}{2\pi i}\int_{\gamma^{1,0}+\gamma^{2,1}+\cdots+\gamma^{k_0,k_0-1}} \frac{f^n(\lambda)\,d\lambda}{\lambda - z}, \qquad z \in \Omega_k-(\gamma^{k,k-1}-\gamma^{k+1,k}). \tag{2}$$

Here $\gamma^{k,k-1}$ is the common boundary of Ω_k with Ω_{k-1}, oriented in the positive direction for the domains from Ω_k; $k_0 = \max k$.

Using now the boundedness of the integral (2) and condition A', we establish that the support γ coincides with the support of $\gamma^{1,0} + \gamma^{2,1} + \cdots + \gamma^{k_0,k_0-1}$. (It follows from this that there are no adjacent domains of the same rank.) The oriented curve γ is either the same as $\gamma^{1,0} + \gamma^{2,1} + \cdots + \gamma^{k_0,k_0-1}$, or opposite to it, i.e.

$$\gamma = \pm(\gamma^{1,0} + \gamma^{2,1} + \cdots + \gamma^{k_0,k_0-1}).$$

Hence the validity of Theorem 3 follows: if $[\varphi, f] = C$, then the complement to $\overline{D}_0$ is connected.

It is now easy to obtain Theorem 2.

Indeed, if

$$\int_{|\zeta|=1} f^n(\zeta)\varphi^m(\zeta)\varphi'(\zeta)\,d\zeta = 0, \qquad n,\ m \geq 0,$$

then this means (since $\varphi'(\zeta) \neq 0$) that $[\varphi, f] \neq C$. Conversely, let $[\varphi, f] \neq C$. Then

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f^n(\zeta)\varphi'(\zeta)\,d\zeta}{\varphi(\zeta) - z} = \frac{1}{2\pi i}\int_{\gamma} \frac{f^n(\lambda)\,d\lambda}{\lambda - z} \equiv 0, \qquad z \in D_0,$$

whence

$$\int_{|\zeta|=1} f^n(\zeta)\varphi^m(\zeta)\varphi'(\zeta)\,d\zeta = 0, \qquad n, m \geq 0.$$

Theorem 4. If $[\varphi, f] \neq C$ and conditions A' and B are satisfied, then

$$\frac{1}{2\pi i}\int_{\pm\gamma}\frac{f^n(\lambda)\,d\lambda}{\lambda - z} = R_k^{(n)}(z) = y_1^n + \dots + y_k^n, \qquad z \in \Omega_k - \gamma,$$

where $|y_1(z)| \leq M$, $R_k^{(n)}(\lambda) - R_{k-1}^{(n)}(\lambda) = f^n(\lambda)$, $\lambda \in \Omega_k \cap \Omega_{k-1}$. Further, if D_k is a component of $\Omega_k - \gamma$, then $y_i(z)$, $z \in D_k$, $i = 1, \dots, k$, are the roots of the equation

$$y^k + C_1(z)y^{k-1} + \dots + C_k(z) = 0,$$

whose coefficients $C_i(z)$ are analytic in D_k and continuous in $\overline{D}_k$.

Now we can prove Theorem 1. We show that if $[\varphi, f] \neq C$, then

$$\Gamma: \quad z_1 = \varphi(\zeta), \quad z_2 = f(\zeta), \quad |\zeta| = 1,$$

bounds a piece of an analytic surface R in the space of two complex variables. The converse fact is known ([2]). As points of $R + \Gamma$ we take the points $(z_0, y_i(z_0))$, where $z_0 \in \Omega_k$, $k \geq 1$, and $y_j(z_0)$ is a root of the equation corresponding to Ω_k. If $z_0 \in \gamma$, and the resultant of the equation

$$y^k + C_1 y^{k-1} + \dots + C_k = 0$$

is not equal to zero at z_0, then in a neighborhood U of the point z_0 one can distinguish k different analytic-

branches of solutions of the equation. Denote by $y_j(z)$ the branch that assumes at z_0 the value $y_j(z_0)$. Then the set of points $(z, y_j(z))$, $z \in U$, forms a neighborhood of the point $(z_0, y_j(z_0)) \in R$, analytically homeomorphic to a disk. If at z_0 the resultant of the equation vanishes, then, since the zeros of the resultant are discrete inside Ω_k, the point $(z_0, y_j(z_0))$ will be a branch point of order $p-1$ (if $y_j(z_0)$ is a root of multiplicity p). Let now $\lambda_0 \in \gamma$, but $\lambda_0 \notin \widetilde{M}$, and let λ_0 lie on the boundary of two domains D_k and D_{k+1} of ranks respectively $k, k+1$, $k \geq 0$. Let $y_1(\lambda_0), \dots, y_r(\lambda_0)$ be the roots of the equation corresponding to D_k and not equal to $f(\lambda_0)$; they coincide with the roots of the equation corresponding to D_{k+1} and not equal to $f(\lambda_0)$. Denote by 3δ the quantity $\min_{1 \leq j \leq r} |f(\lambda_0) - y_j(\lambda_0)|$. Then, by continuity of the coefficients of the equation in $\overline{D}_k$ (and in $\overline{D}_{k+1}$), there exists a neighborhood V of the point λ_0 in which the roots of the equation differ by less than δ from $y_j(\lambda_0)$, $1 \leq j \leq k$. At every point $z_1 \in V \cap D_k$ at which the resultant of the equation is nonzero, there are exactly r roots (we shall call them roots of the first group) for which $|y_j(z_1) - f(\lambda_0)| > 2\delta$, $1 \leq j \leq r$, and $k - r$ roots for which $|y_j(z_1) - f(\lambda_0)| < \delta$, $r < j \leq k$. Selecting in a neighborhood of

z_1 analytic branches and carrying out analytic continuation inside $V \cap D_k$ to the point $z_2 \in V \cap D_k$, we again pass from roots of the first group to roots of the first group. Hence it follows that the symmetric functions formed from the roots of the first group, $C_1' = -(y_1 + \cdots + y_r), \ldots, C_r' = (-1)^r y_1 \cdots y_r$, are single-valued analytic bounded functions in $V \cap D_k$ (and in $V \cap D_{k+1}$) and coincide on $V \cap \gamma$. Consequently they are analytic in V, and the resultant of the equation $y^r + C_1'(z) y^{r-1} + \cdots + C_r'(z) = 0$ has inside V a discrete number of zeros. Thus, if $y_j(\lambda_0) = f(\lambda_0)$, then $(\lambda_0, y_j(\lambda_0)) \in \Gamma$; if $y_j(\lambda_0) \neq f(\lambda_0)$, then $(\lambda_0, y_j(\lambda_0))$ is an algebraic branch point of R (in particular, a regular point). The case when $\lambda_0 \in \widetilde{M}$ is investigated similarly. For every $\lambda \in \gamma$ that is a boundary point of several domains, one of the roots of the equations corresponding to these domains is equal to $f(\lambda)$; thus the boundary of R is the whole curve Γ. Since Γ is a simple curve, R is connected. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.