MATHEMATICS

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Abstract

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PERIODIC SOLUTION OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVES*

1. We shall consider the system of differential equations

$$\varepsilon dx/dt = f(x,y), \qquad dy/dt = g(x,y), \tag{1}$$

where $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_l)$ are vectors, $\varepsilon > 0$ is a small parameter; the vector functions $f(x,y) = (f_1, \dots, f_k)$, $g(x,y) = (g_1, \dots, g_l)$ are assumed to be three times continuously differentiable in the domain under consideration. With respect to the system of fast motions

$$dx/d\tau = f(x,y) \qquad (y = \text{const—parameter}) \tag{2}$$

it is assumed that, for any y, it has exactly one nondegenerate periodic solution $x^*(\tau, y)$ with period $T(y)$. In other words, the multipliers of the system of variational equations**

$$\dot{\xi} = f_y(x,y)\big|_{x=x^*(\tau,y)}\xi \tag{3}$$

are different from unity, except for one; we denote these multipliers by $\lambda_1, \dots, \lambda_{k-1}$. It is natural to introduce the "averaged system"

$$\frac{dy}{dt} = \bar{g}(y) = \frac{1}{T(y)}\int_0^{T(y)} g(x^*(\tau,y),y)\,d\tau = \int_0^1 g[X(\varphi,y),y]\,d\varphi, \tag{4}$$

where $X(\varphi, y) = x^*(T(y)\varphi, y)$ is put. With respect to the averaged system (4) we shall assume that it has a nondegenerate equilibrium position y_0; in other words, $\bar{g}(y_0) = 0$ and the eigenvalues $\mu_1, \dots, \mu_l$ of the matrix $\bar{g}_y(y_0)$ are all different from zero.

Our aim is the following

Theorem. *If ε is sufficiently small, then, under the assumptions made, system (1) has, in a neighborhood of the cycle $\{x^*(\tau, y_0), y_0\}$, a unique periodic solution $\{x(t,\varepsilon), y(t,\varepsilon)\}$ with the following properties: its period is equal to $\varepsilon T(y_0)+O(\varepsilon^2)$, and $|y(t,\varepsilon)-y_0| = O(\varepsilon)$. Moreover, there exists a function $\varphi(t,\varepsilon)$ ("phase") depending smoothly on t such that*

$$|\varepsilon d\varphi/dt - 1/T(y_0)| = O(\varepsilon), \qquad |x(t,\varepsilon) - X(\varphi(t,\varepsilon), y_0)| = O(\varepsilon), \tag{5}$$

*and the multipliers of the variational equation for this periodic solution that are different from unity are****

$$\nu_i = \lambda_i + \omega(\varepsilon) \qquad (i = 1, \dots, k-1); \tag{6}$$

$$\nu_{j+k-1} = 1 + \varepsilon\mu_{jT}(y_0) + o(\varepsilon) \qquad (j = 1, \dots, l).$$

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* The principal results of the present work were reported at the Third All-Union Mathematical Congress (1).

** Here $f_y(x,y)$ is the matrix $\|\partial f_i(x,y)/\partial y_j\|$. Analogous notation is used below as well.

*** $\omega(\varepsilon)$ denotes a quantity tending to 0 as $\varepsilon \to 0$. A careful examination of the proof shows that, if λ_i or μ_j is not a multiple root, then $\omega(\varepsilon)$, respectively $o(\varepsilon)$, may be replaced by $O(\varepsilon)$, respectively $O(\varepsilon^2)$.

2. By a nondegenerate transformation of the form

$$\xi = \frac{\partial x^*}{\partial \tau} u_0 + A\left(\frac{\tau}{T(y)}, y\right) u$$

(u_0 is a scalar, u is a vector with $k-1$ components, $A(\varphi, y)$ is a matrix, continuously differentiable three times, with k rows and $k-1$ columns, having period 1 in φ), system (3) can be brought to the form

$$du_0/d\tau = 0, \qquad du/d\tau = H(\tau/T(y), y)u, \tag{7}$$

where $H(\varphi, y) = H(\varphi+1, y)$ is a twice continuously differentiable square matrix of order $k-1$, and the multipliers of the system $\dot{u} = Hu$ are our $\lambda_1, \dots, \lambda_{k-1}$. System (1), by the change of variables $x = X(\varphi, y) + A(\varphi, y)u$, is brought to the form

$$\varepsilon\, d\varphi/dt = 1/T(y) + P(\varepsilon, \varphi, u, y),$$

$$\varepsilon\, du/dt = H(\varphi, y)u + \varepsilon b(\varphi, y) + Q(\varepsilon, \varphi, u, y), \tag{8}$$

$$dy/dt = g[X(\varphi, y), y] + R(\varphi, u, y),$$

where P, Q, R, b are smooth functions of their arguments, with $R(\varphi, u, y)$ and $R_y = O(|u|)$, $Q(\varepsilon, \varphi, u, y) = O(\varepsilon^2 + u^2)$, $P(\varepsilon, \varphi, u, y)$, Q_u, and $Q_y = O(\varepsilon + |u|)$ uniformly in φ, y.

Let us take φ as the independent variable. System (8) becomes

$$du/d\varphi = T(y)H(\varphi, y)u + \varepsilon T(y)b(\varphi, y) + \widetilde{Q}(\varepsilon, \varphi, u, y),$$

$$dy/d\varphi = \varepsilon h(\varphi, y) + \varepsilon \widetilde{R}(\varepsilon, \varphi, u, y), \tag{9}$$

where $h(\varphi, y) = T(y)g[X(\varphi, y), y]$, and $\widetilde{Q}, \widetilde{R}$ are smooth functions of their arguments, with $\widetilde{Q}(\varepsilon, \varphi, u, y) = O(\varepsilon^2 + u^2)$, Q_u, Q_y, R, and $R_y = O(\varepsilon + |u|)$ uniformly in φ, y. Anticipating what follows, we introduce also

$$\bar{h}(y) = T(y)\bar{g}(y) = \int_0^1 h(\varphi, y)\, d\varphi.$$

As follows directly from the first of equations (8), a periodic solution of system (8) lying near $\{x^*(\tau, y_0), y_0\}$, whose period is close to $\varepsilon T(y_0)$, is transformed, in passing from (8) to (9), into a solution of system (9) with period 1 in φ. Denote the solution of system (9) with initial value $u = u_n, y = y_n$ at $\varphi = 0$ by $\{u(u_n, y_n, \varphi, \varepsilon), y(u_n, y_n, \varphi, \varepsilon)\}$. It will be periodic in φ with unit period if the conditions

$$u^*(u_n, y_n, \varepsilon) = u(u_n, y_n, 1, \varepsilon) - u_n = 0,$$

$$\hat{y}(u_n, y_n, \varepsilon) = y(u_n, y_n, 1, \varepsilon) - y_n = 0 \tag{10}$$

are fulfilled. Since

$$\hat{y} = \varepsilon \int_0^1 T(y(\varphi))g(x(\varphi), y(\varphi))\, d\varphi,$$

it is natural, instead of (10), to introduce the system

$$u^*(u_n, y_n, \varepsilon) = 0, \qquad y^* = \frac{1}{\varepsilon}\hat{y}(u_n, y_n, \varepsilon) = 0. \tag{11}$$

We shall verify that system (11) is satisfied for $\varepsilon = 0,\ u_n = 0,\ y_n = y_0$, and that the Jacobian

$$J = D(u^*, y^*)/D(u_n, y_n)\big|_{\varepsilon=0,\ u_n=0,\ y_n=y_0} \neq 0.$$

For $\varepsilon = 0$ we have $y = \text{const}$ and

$$du/d\varphi = T(y)H(\varphi, y)u + \widetilde{Q}(0, \varphi, u, y), \tag{12}$$

where $\widetilde{Q}(0, \varphi, u, y) = O(u^2)$, $\widetilde{Q}_u(0, \varphi, u, y) = O(|u|)$. Therefore, for any y_n, $u = 0$ is a solution of (12). Hence, further, it follows that

$$\left.\frac{D(u^*)}{D(y_n)}\right|_{\varepsilon=0,\ u_n=0,\ y_n=y_0} = 0,$$

so that

$$J = \left.\frac{D(u^*)}{D(u_n)}\,\frac{D(y^*)}{D(y_n)}\right|_{\varepsilon=0,\ u_n=0,\ y_n=y_0}.$$

Since

$$y^*(0, y_n, 0) = \int_0^1 h(\varphi, y_n)\, d\varphi = T(y_n)\overline{g}(y_n),$$

we have $y^*(0, y_0, 0) = 0$, and $D(u^*)/D(u_n)|_{\varepsilon=0,\ y=y_n}$ is the determinant of the matrix $T(y_n)\overline{g}(y_n) + T_y(y_n)\overline{g}(y_n)$. But for $y_n = y_0$ the second term here vanishes, and

$$\left.\frac{D(u^*)}{D(u_n)}\right|_{\varepsilon=0,\ u_n=0,\ y_n=y_0} = \det[T(y_0)\overline{g}_y(y_0)] \neq 0.$$

Finally, the Jacobian

$$\left.\frac{D(u^*)}{D(u_n)}\right|_{\varepsilon=0,\ u_n=0,\ y_n=y_0} \neq 0,$$

since it is the determinant of the difference between the monodromy matrix for the system $du/d\varphi = T(y_0)H(\varphi, y_0)u$ and the identity matrix.

Consequently, for sufficiently small ε, system (11) determines functions $u_n(\varepsilon)$ and $y_n(\varepsilon)$ having, at the point $\varepsilon = 0$, bounded derivatives with respect to ε. It is not hard to verify that the periodic solution of system (8) with initial value $u_n(\varepsilon), y_n(\varepsilon)$ has the required properties; only (6) requires a separate proof.

3. To compute the multipliers ν_i $(i = 1, \dots, l + k - 1)$ of the variational equations for the periodic solution found by us, we consider the variational equations for the periodic solution $\{u_\varepsilon(\varphi), y_\varepsilon(\varphi)\} = \{u_\varepsilon(\varphi+1), y_\varepsilon(\varphi+1)\}$ of system (9). Since $y_\varepsilon(\varphi) = y_0 + O(\varepsilon)$, $u_\varepsilon(\varphi) = O(\varepsilon)$, these variational equations have the form

$$\frac{d}{d\varphi}\delta u = [T(y_0)H(\varphi, y_0) + O(\varepsilon)]\delta u + O(\varepsilon)\delta y, \tag{13}$$

$$\frac{d}{d\varphi}\delta y = \varepsilon B(\varphi, \varepsilon)\delta u + \varepsilon[h_y(\varphi, y_0) + O(\varepsilon)]\delta y, \tag{14}$$

$(B(\varphi, \varepsilon) = B(\varphi + 1, \varepsilon)$ is a continuous matrix function.)

Of course, it is still necessary to show that the multipliers do not change in passing from (8) to (9). The simplest way to see this is to use the following circumstances. It is not hard to check that, for a periodic solution $\{x(t)\}$ of any system $\dot{x} = f(x)$ $(x = (x_1, \dots, x_m))$, the multipliers (with the exception of the multiplier 1, to which the solution $f(x(t))$ of the variational equations corresponds) may be defined as follows: take any $(m—1)$-dimensional hyperplane Π intersecting $\{x(t)\}$ at some point x_0, with $f(x_0)$ not parallel to Π, and consider its successive mapping F, which assigns to a point $x \in \Pi$ the first, in time, point of intersection with Π of the positive semitrajectory $t \geq 0$ of the system $\dot{x} = f(x)$ issuing from x; F is defined in some neighborhood of the point x_0 on Π, and the multipliers of $\{x(t)\}$ coincide with the eigenvalues of the differential of the mapping F at the point x_0. But the mapping F depends only on the geometric arrangement of the trajectories and on the direction of motion along them, and not on the speed of this motion.

Now it is convenient to introduce the new variable

$$z = \left\{E - \varepsilon\int_0^\varphi [h_y(\theta, y_0) - \overline{h}_y(y_0)]\, d\theta\right\}\delta y$$

(E is the identity matrix). System (13)–(14) will take the following form: equation (13) is preserved with the replacement of $O(\varepsilon)\delta y$ by $O(\varepsilon)z$, and instead of (14) we obtain

$$dz/d\varphi = \varepsilon B(\varphi, \varepsilon)\delta u + \varepsilon[\overline{h}_y(y_0) + O(\varepsilon)]z. \tag{15}$$

The matriciant of the system

$$d\xi/d\varphi = T(y_0)H(\varphi, y_0)\xi, \qquad d\zeta/d\varphi = \varepsilon[B(\varphi, \varepsilon)\xi + \tilde{h}_y(y_0)\zeta]$$

has the form

$$\Phi(\varphi) = \begin{pmatrix} \Phi_1(\varphi) & 0 \\ \Phi_2(\varphi) & \Phi_3(\varphi) \end{pmatrix},$$

where the matrices $\Phi_1(1)$ and $\Phi_3(1)$ have eigenvalues λ_i $(i = 1, \dots, k-1)$ and $e^{\varepsilon \mu_j T(y_0)}$ $(j = 1, \dots, l)$, respectively, while $\Phi_2(1) = O(\varepsilon)$. We seek the matriciant of the system (13), (15) in the form $\Psi(\varphi) = \Phi(\varphi)[E + \Delta(\varphi)]$. It is not difficult to verify that $\Delta(\varphi)$ satisfies an equation of the form

$$\frac{d\Delta}{d\varphi} = \begin{pmatrix} O(\varepsilon) & O(\varepsilon) \\ O(\varepsilon^2) & O(\varepsilon^2) \end{pmatrix} (E + \Delta); \qquad \Delta(0) = 0,$$

whence it follows that

$$\Delta(1) = \begin{pmatrix} O(\varepsilon) & O(\varepsilon) \\ O(\varepsilon^2) & O(\varepsilon^2) \end{pmatrix},$$

and therefore

$$\Psi(1) = \begin{pmatrix} \Phi_1(1) + O(\varepsilon) & O(\varepsilon) \\ \Phi_2(1) + O(\varepsilon^2) & \Phi_3(1) + O(\varepsilon^2) \end{pmatrix}.$$

Now it is immediately clear that $\Psi(1)$ has $k-1$ eigenvalues $\nu_i = \lambda_i + \omega(\varepsilon)$ $(i = 1, \dots, k-1)$. The situation is more complicated with the remaining eigenvalues. It is directly obvious only that $\nu_{j+k-1} = 1 + \omega(\varepsilon)$ $(j = 1, \dots, l)$. Subtract from $\Psi(1)$ the identity matrix of order $k + l - 1$; we obtain a matrix of the form

$$\mathfrak{A}(\varepsilon) = \begin{pmatrix} A + F(\varepsilon) & B(\varepsilon) \\ C(\varepsilon) & \varepsilon D + G(\varepsilon) \end{pmatrix},$$

where F, B, and $C = O(\varepsilon)$, $G(\varepsilon) = O(\varepsilon^2)$, and A and D are nonsingular matrices with eigenvalues $\lambda_1 - 1, \dots, \lambda_{k-1} - 1$ and $T(y_0)\mu_1, \dots, T(y_0)\mu_l$, respectively. We must prove that, in addition to the obvious eigenvalues $\nu_i = \lambda_i - 1 + \omega(\varepsilon)$ $(i = 1, \dots, k-1)$, $\mathfrak{A}(\varepsilon)$ also has eigenvalues

$$\nu_{j+k-1} = \varepsilon T(y_0)\mu_j + o(\varepsilon) \qquad (j = 1, \dots, l). \tag{16}$$

Let K be an arbitrary matrix with $k-1$ rows and l columns; put $C(K, \varepsilon) = C(\varepsilon) - KB(\varepsilon)K - \varepsilon DK - G(\varepsilon)K$. For arbitrary K, K', the norm

$$|C(K, \varepsilon) - C(K', \varepsilon)| \leq O(\varepsilon)(1 + |K| + |K'|)|K - K'|,$$

and it is easy to prove that, for small ε, the equation

$$K = -C(K,\varepsilon)(A + F(\varepsilon))^{-1}$$

has a solution $K(\varepsilon) = O(\varepsilon)$. (16) now follows from

$$\begin{pmatrix} E & 0 \\ K(\varepsilon) & E \end{pmatrix} \mathfrak{A}(\varepsilon) \begin{pmatrix} E & 0 \\ K(\varepsilon) & E \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} A + F - BK & B \\ K(A+F) + C(K,\varepsilon) & \varepsilon D + G + KB \end{pmatrix} = \begin{pmatrix} A + O(\varepsilon) & O(\varepsilon) \\ 0 & \varepsilon D + O(\varepsilon^2) \end{pmatrix}.$$

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References

1. L. S. Pontryagin, *Proceedings of the Third All-Union Mathematical Congress*, 2, Moscow, 1956, p. 93; 3, Moscow, 1958, p. 570.

Note: Figure translations are in progress. See original paper for figures.

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