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Abstract

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MATHEMATICS

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ON SOME PROBLEMS IN THE TOPOLOGY OF MANIFOLDS CONNECTED WITH THE THEORY OF THOM SPACES

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In this paper we consider smooth manifolds W^i situated in Euclidean space R^{n+i}, whose normal bundle has as its structure group some subgroup G of the group $O(n)$. On the set of all such manifolds an equivalence relation is introduced. The set $V_n^i(G)$ of equivalence classes of manifolds forms an abelian group. The equivalence relation and the group operation are introduced on the set of our manifolds literally in the same way as in Pontryagin' s classical interpretation of the homotopy groups of spheres by means of framed manifolds, and in the well-known construction of the Thom–Rokhlin inner homology groups. Following L. S. Pontryagin, we shall call a manifold W^i, embedded in Euclidean space R^{n+i}, G-framed if in the normal bundle of the manifold W^i a fixed structure of a G-bundle is given. A G-framed manifold $W^i \subset R^{n+i}$ is called equivalent to zero if there exists a smooth manifold $N^{i+1} \subset R^{n+i+1}$ with boundary $W^i \subset R^{n+i}$, such that the normal G-bundle can be extended from the manifold W^i to the manifold N^{i+1}. It is evident that for any groups $G_1 \subset O(n_1)$ and $G_2 \subset O(n_2)$ the operation of direct multiplication of manifolds induces a certain pairing

$$V_{n_1}^i(G_1) \otimes V_{n_2}^j(G_2) \to V_{n_1+n_2}^{i+j}(G_1 \times G_2) \tag{1}$$

(the group $O(n_1) \times O(n_2)$ is regarded in the known way as embedded in the group $O(n_1 + n_2)$). We shall restrict ourselves here to the study of the groups $V_n^i(SO(n))$ for $i < n-1$, the groups $V_{2n}^i(u(n))$ for $i < 2n-2$, and the groups $V_{4n}^i(\mathrm{Sp}(n))$ for $i < 4n-4$. It is easy to see that these groups do not depend on n. We shall denote them respectively by the symbols V_{SO}^i, V_u^i, V_{Sp}^i. Obviously, the above pairing allows one to define the direct sums $V_{SO} = \sum V_{SO}^i$, $V_u = \sum V_u^i$, $V_{\mathrm{Sp}} = \sum V_{\mathrm{Sp}}^i$ as graded rings.

Theorem 1. *The quotient ring of the ring V_{SO} by its 2-torsion is isomorphic to the polynomial ring on generators V_{4i} of dimension $4i$, $i \geq 0$. The ring V_u is isomorphic to the polynomial ring on generators u_{2i} of dimension $2i$, $i \geq 0$.*

The algebras $V_{\mathrm{Sp}} \otimes Z_p$, $p > 2$, and $V_{\mathrm{Sp}} \otimes Q$, where Q is the field of rational numbers, are isomorphic to polynomial algebras on generators t_{4i} of dimension $4i$, $i \geq 0$. The rings V_{SO}, V_u, V_{Sp} have no p-torsion for $p > 2$, and the ring V_u has no 2-torsion.

This theorem for the ring V_{SO} was announced by Milnor ([5]). The structure of the groups V_{SO}^i was studied in ([1],[4]) by other methods. For the proof of the theo-

of Theorem 1, let us first note that, using the known constructions of Thom ([3]), one can without difficulty prove the isomorphisms $V_n^i(G) \approx \pi_{n+i}(M_n(G))$, where $M_n(G)$ is the space constructed by Thom for the study of G-realizations of cycles. Moreover, one can prove that the pairing

$$\pi_{n_1+i}(M_{n_1}(G)) \otimes \pi_{n_2+j}(M_{n_2}(G)) \to \pi_{n_1+n_2+i+j}(M_{n_1+n_2}(G_1 \times G_2)),$$

corresponding, by virtue of these isomorphisms, to the pairing (1), is induced by the natural homeomorphism

$$\rho: \quad M_{n_1}(G_1) \times M_{n_2}(G)/M_{n_1}(G_1) \vee M_{n_2}(G) \to M_{n_1+n_2}(G_1 \times G_2).$$

Thus, the study of the groups $V_n^i(G)$ and of the pairing (1) reduces to the study of the homotopy properties of the spaces $M_n(G)$. By analogy with the groups $V_{SO}, V_u, V_{\mathrm{Sp}}$ we introduce the graded groups $H_{SO}(p)$, $H_u(p)$, and $H_{\mathrm{Sp}}(p)$, whose homogeneous components are the stable cohomology groups over the field Z_p of the corresponding Thom spaces. Obviously, these groups may be regarded as modules over the Steenrod algebra $A = A_p$.

Lemma 1. *For $p > 2$, the module $H_{SO}(p)$ admits a system of generators u_ω, in one-to-one correspondence with partitions ω of numbers divisible by four into summands divisible by four and not of the form $2p^t - 2$. The dimension of the generator u_ω is equal to the sum of the terms of the partition ω. For any element $x \in H_{SO}(p)$ the relation $\beta x = 0$ holds, where β is the Bockstein homomorphism. All nontrivial relations between elements of the module $H_{SO}(p)$ follow from this relation. The module $H_{\mathrm{Sp}}(p)$, for $p > 2$, is isomorphic to $H_{SO}(p)$. The module $H_u(p)$, for any $p \geq 2$, is described in the same way as the module $H_{SO}(p)$ for $p > 2$, with the only difference that partitions of arbitrary even numbers into even summands not of the form $2p^t - 2$ are allowed. The module $H_{\mathrm{Sp}}(2)$, as for $p > 2$, admits a system of generators u_ω, but the condition that the summands of the partitions ω are not of the form $2p^t - 2$ is replaced by the condition that these summands are not of the form $4(2^t - 1)$. Moreover, for any element $x \in H_{\mathrm{Sp}}(2)$, besides the relation $\beta x = \mathrm{Sq}^1 x = 0$, the relation $\mathrm{Sq}^2 x = 0$ also holds. All nontrivial relations between elements of the module $H_{\mathrm{Sp}}(2)$ follow from these relations. The module $H_{SO}(2)$ decomposes into a direct sum of a free module and modules M_ω with one generator u_ω, where ω is an arbitrary partition of a number divisible by four into summands divisible by four. The dimension of* *the generator u_ω is equal to the sum of the terms of the partition ω. The only nontrivial relation in the module M_ω is the relation $\mathrm{Sq}^1 u_\omega = 0$.*

It can further be proved that the mappings

$$H_{SO}(p) \to H_{SO}(p) \otimes H_{SO}(p),$$

$$H_u(p) \to H_u(p) \otimes H_u(p),$$

$$H_{\mathrm{Sp}}(p) \to H_{\mathrm{Sp}}(p) \otimes H_{\mathrm{Sp}}(p),$$

induced by the homeomorphism ρ, are expressed by the formula

$$\rho^*(u_\omega) = \sum_{\substack{(\omega_1,\omega_2)=\omega \\ \omega_1 \neq \omega_2}} [u_{\omega_1} \otimes u_{\omega_2} + u_{\omega_2} \otimes u_{\omega_1}] + \sum_{(\omega_1,\omega_1)=\omega} u_{\omega_1} \otimes u_{\omega_1}.$$

By virtue of Lemma 1, the study of the modules $H_{SO}(p)$ and $H_{\mathrm{Sp}}(p)$ for $p > 2$, and of the module $H_u(p)$ for $p \geq 2$, reduces to the study of the module M_β with one generator

(the dimension of which we shall regard as equal to zero), defined by the relation $\beta x = 0$ for all $x \in M_\beta$. Since, as is easy to see, for the module M_β there is defined a diagonal mapping $M_\beta \to M_\beta \otimes M_\beta$, the group

$$\mathrm{Ext}_A(M_\beta, Z_p) = \sum \mathrm{Ext}_A^{s,t}(M_\beta, Z_p)$$

is defined as an algebra. Relying on the bases of the Steenrod algebra constructed by Adams ([2]), one can prove that the following holds:

Lemma 2. The algebra $\mathrm{Ext}_A(M_\beta, Z_p)$ is isomorphic to the polynomial algebra in the generators $h_t \in \mathrm{Ext}_A^{1,\,2p^t-1}(M_\beta, Z_p)$, $p \geq 2,\ t \geq 0$.

Theorem 1 now follows without difficulty from Lemmas 1 and 2 and the properties of the Adams spectral sequence ([2]). An essential supplement to Theorem 1 is

Theorem 2. The factor ring of the ring V_{Sp} with respect to 2-torsion is not isomorphic to a polynomial ring. Namely, there exist generators $x \in V_{\mathrm{Sp}}^4$ and $y \in V_{\mathrm{Sp}}^8$ of infinite order such that $2^k(x^2 - 4y) = 0$.

The proof of Theorem 2 is carried out analogously to the proof of Theorem 1. Here, in view of Lemma 1, instead of the module M_β one should consider the module M_1 with one generator, defined by the relation $\mathrm{Sq}^1 z = \mathrm{Sq}^2 z = 0$ for all

$z \in M_1$. For this case the group $\operatorname{Ext}_A(M_1, Z_2)$ is also an algebra, and Lemma 2 is replaced by the following Lemma 2′:

Lemma 2′. The algebra $\operatorname{Ext}_A(M_1, Z_2) = \sum \operatorname{Ext}_A^{s,t}(M_1, Z_2)$ is isomorphic to the cohomology algebra of a certain algebra B, possessing the following properties:
1) the algebra B contains a central subalgebra C, admitting a system of generators $\alpha_{r,0} \in C^{(2^r-1)}$, $r \geq 2$, satisfying only the relations $\alpha_{r,0}^2 = 0$ (and their consequences);
2) the algebra $B//C$ is commutative and admits a system of generators $\alpha_0 \in B//C^{(1)}$, $\alpha_{r,1} \in B//C^{(2^{r+1}-2)}$, $r \geq 1$, satisfying only the relations $\alpha_0^2 = 0$ and $\alpha_{r,1}^2 = 0$;
3) in the Serre-Hochschild spectral sequence of the pair (B, C) the following relations hold:

$$d_2(1 \otimes h_{r,0}) = h_0 h_{r-1,1} \otimes 1,$$

$$d_3(1 \otimes h_{r,0}^2) = h_{1,1} h_{r-1,1}^2 \otimes 1,$$

$$d_i(1 \otimes h_{r,0}^4) = 0, \qquad i \geq 2,$$

where $h_{r,0}$ are the generators of the algebra $H^*(C)$, determined by the equality $(h_{r,0}, \alpha_{r,0}) = 1$, and h_0 and $h_{r,1}$ are generators of the algebra $H^*(B//C)$, determined respectively by the equalities $(h_0, \alpha_0) = 1$ and $(h_{r,1}, \alpha_{r,1}) = 1$.

Since Theorems 1 and 2 are, in essence, theorems on the homotopy groups of Thom spaces, they can be applied to the problem of realizing integral cycles of manifolds as submanifolds.

Theorem 3. An integral homology class Z_{n-i} of a compact, closed, orientable, smooth manifold M^n is realized by a submanifold $W^{n-i} \subset M^n$ if $i > [n/2] + 1$ and, for $k < n - i - 2(p-1)$, the groups $H_k(M^n)$ have no p-torsion for all $p > 2$. An integral homology class Z_{n-2i} of the manifold M^n admits a $U(i)$-realization if $2i > [n/2] + 1$ and, for $k < n - 2i - 2(p-1)$, the groups $H_k(M^n)$ have no p-torsion for all $p \geq 2$.

Let K be an arbitrary finite polyhedron. From Theorem 3 there follows

Corollary. If the groups $H_q(K)$ for $q > i - 2(p-1)$ have no p-torsion for all $p > 2$, then every cycle $Z_i \in H_i(K)$ can be represented in the form of the continuous image of the fundamental cycle of some orientable manifold W^i.

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Note: Figure translations are in progress. See original paper for figures.