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Abstract

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ON THE SPECTRAL THEORY OF LINEAR OPERATORS IN A BANACH SPACE

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Let A be a linear operator in a Banach space. We shall agree to call it an S-operator if the following requirements are satisfied:

I. The spectrum of the operator A lies on the real axis.

II. For every finite interval Δ of the real axis there exists a subspace $L(\Delta)$, invariant with respect to the operator A, such that: a) on $L(\Delta)$ the operator A is defined everywhere and is bounded; b) the spectrum of the part of the operator A induced on $L(\Delta)$ consists of the intersection of the spectrum of the operator A with the interval Δ and, possibly, the endpoints of the interval Δ; c) every invariant subspace on which the operator A is defined everywhere, is bounded, and has as spectrum a part of the segment Δ, is contained in $L(\Delta)$.

III. The system of invariant subspaces $L(\Delta)$, corresponding to any covering of the real axis by finite intervals, is complete.

It is obvious that every S-operator is defined on a dense domain, is closed, and its spectrum is nonempty.

Examples of S-operators may be a self-adjoint operator, and also a bounded operator defined on the whole space with real nowhere dense spectrum.

Let us give an example of a bounded operator defined on the whole space, with real spectrum, which is not an S-operator.

Take a sequence $\{\mu_n\}_0^\infty$ of positive numbers, monotonically tending to $+\infty$, for which

$$\sum_{n=1}^{\infty} \frac{1}{n\mu_n} = \infty,$$

and denote by $Q\{\mu_n\}$ the Hilbert space of infinitely differentiable functions $x(s)$ $(0 \leqslant s \leqslant 1)$ satisfying the condition

$$\|x\| \equiv \left[\sum_{n=0}^{\infty} \frac{1}{n!^2 \mu_n^{2n}} \int_0^1 |x^{(n)}(s)|^2 \, ds \right]^{1/2} < \infty.$$

By a known theorem of T. Carleman (^{1}), the space $Q\{\mu_n\}$ is a quasi-analytic class. The operator of multiplication by the independent variable s in the space $Q\{\mu_n\}$ has all the required properties.

In the present note some conditions are given for an operator to be an S-operator. Related questions were studied in the works (^{2}).

We shall call a linear operator A in a Banach space **locally correct** if: 1) A is a closed operator with dense domain of definition D_A; 2) the Cauchy problem

$$i\frac{dx(t)}{dt} = Ax(t) \quad (-\infty < t < \infty),$$

$$x(0) = x$$

for every $x \in D_A$ has a unique solution

$$x(t) = U_t x$$

in the class of strongly differentiable vector-functions; 3) the operator U_t is uniformly bounded on every finite interval of variation of t.

An everywhere defined bounded operator is, of course, locally correct. In general, for local correctness of a closed operator with dense domain it is necessary and sufficient that it be the infinitesimal generator of a strongly continuous group of bounded operators (^{3}).

Theorem 1. *If the operator A is locally correct and if the integral* *

$$\int_{-\infty}^{\infty} \frac{\ln \|U_t\|}{1+t^2} \, dt, \tag{1}$$

converges, then A is an S-operator.

Underlying this theorem is the idea of the equivalence of the spectral analysis of the operator A and the harmonic analysis of the functions $f(U_t x)$ (x is an arbitrary vector, f an arbitrary linear functional). Passing to one or another evolution equation for the purpose of investigating the spectral properties of an operator has been used repeatedly in the theory of differential operators (^{4}).

The convergence condition for the integral (1) is known in the theory of normed rings as the regularity condition of the ring (^{5}) generated by the operator A.

In our case it appears directly in the harmonic analysis of "rapidly increasing" functions $f(U_t x)$ as the condition of quasi-analyticity of the Fourier transforms of the corresponding "basic" functions (^{6}).

The constructions used in the proof of Theorem 1 are based on certain results of V. A. Marchenko and O. I. Inozemtsev (^{7}) and L. I. Ronkin (^{8}).

The convergence condition for the integral (1) in the formulation of Theorem 1 cannot be weakened. Namely, the following is true.

Theorem 2. *Let a measurable locally bounded function* $\alpha(t) \geq 1$ $(-\infty < t < \infty)$ *satisfy the condition*

$$\alpha(t+s) \leq \alpha(t)\alpha(s) \qquad (-\infty < t, s < \infty).$$

If, moreover,

$$\int_{-\infty}^{\infty} \frac{\ln \alpha(t)}{1+t^2}\, dt = \infty,$$

then there exists a Hilbert space and in it a locally correct operator A *with real spectrum, which is not an* S*-operator, such that*

$$\|U_t\| \leq \alpha(t) \qquad (-\infty < t < \infty).$$

The operator A mentioned, in the most important case, when

$$\lim_{t \to \pm\infty} \frac{\ln \alpha(t)}{t} = 0,$$

is constructed as the differentiation operator $-i\,\dfrac{d}{ds}$ in the class $L^2_\alpha(-\infty, \infty)$.

This operator has not a single invariant subspace on which

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[*] The question of the character of convergence of the integral (1) does not arise, since it can be shown that in the case of convergence one always has $\|U_t\| \geq 1$.

on which it would be defined everywhere and bounded. The reason for this is the quasianalyticity of Fourier transforms of functions from $L^2_{a^2}(-\infty, \infty)$.

For an everywhere-defined bounded operator A, convergence of the integral (1) is equivalent to the reality of the spectrum and to the convergence, in this case, of the integral

$$\int_0 \ln \ln M(\delta)\, d\delta, \tag{2}$$

where

$$M(\delta) = \sup_{|\operatorname{Im}\lambda| \geq \delta} \|R_\lambda\|,$$

R_λ is the resolvent of the operator. There exists, however, also a direct method of spectral analysis of operators with real spectrum and with convergent integral (2). This method is based on a theorem of N. Levinson ([9], p. 135), which makes it possible to construct analytic functions with certain special properties.

Theorem 3. *Let A be a closed operator, defined on a dense domain, with real spectrum. If*

$$\int_0 \ln\ln M(\delta)\, d\delta < \infty, \tag{3}$$

then A is an S-operator.

This theorem, in applications, is apparently considerably more effective than Theorem 1.

The technique developed in the proof of Theorem 3 carries over, without substantial complications, to operators whose spectrum, in a neighborhood of each of its points of accumulation, lies on some analytic arc. This technique also makes it possible to separate parts of the spectrum connected with one another by a finite number of analytic arcs. Of course, in this case conditions analogous to condition (3) must be satisfied.

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Note: Figure translations are in progress. See original paper for figures.