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Abstract

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MATHEMATICS

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ON GENERAL SYSTEMS OF DIFFERENTIAL EQUATIONS

(Presented by Academician M. V. Keldysh, 31 XII 1959)

It is well known that the properties of a single differential equation are determined, in the main (with the exception of the so-called degenerate cases), by its principal part, i.e., by the terms of the equation containing the highest derivatives. In studying systems of equations it is also sometimes necessary to single out the principal terms of the system. However, in the case of a system of differential equations, the selection of the principal terms of the system is not as unambiguous as in the case of a single equation, and the "principal part" of the system can be defined in different ways. We note that precisely different ways of selecting the principal part of a system correspond to different definitions of ellipticity (^{1,2}) and hyperbolicity (^{3,4}). In the first part of the present paper we shall indicate a certain method for constructing the principal part of a system and give a definition of the characteristic form for a system of equations. Using the properties of the characteristic form, we shall indicate an effective definition of ellipticity equivalent to the definition of Douglis and Nirenberg (^{2}).

Further, general systems of ordinary equations will be considered specially, and a method will be indicated for reducing them to systems whose principal part is a diagonal matrix. This reduction will enable us to determine the number of arbitrary constants in the general solution of a system of ordinary equations with variable coefficients and to construct a fundamental system of solutions for this system.

I. Consider the system of equations

$$F_i = \sum_{j=1}^{m} A_{ij}(x, D) u_j(x) = f_i(x), \qquad i = 1, \dots, m; \tag{1}$$

$$A_{ij}(x, D) = \sum_{|\alpha| \leq \gamma_{ij}} a_{ij}^{(\alpha)}(x) D^{\alpha}, \tag{2}$$

where, as usual, $x = (x_1, \dots, x_n)$, $D = (D_1, \dots, D_n)$, $D_k = \partial/\partial x_k$; $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $\alpha_1, \dots, \alpha_n$

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are nonnegative integers. The numbers γ_{ij} are called the orders of the operators A_{ij}.

To the system (1) we shall associate the matrix of orders $\Gamma = (\gamma_{ij})$. If $A_{ij} = 0$, then we shall assume that $\gamma_{ij} = -\infty$. Put $R_T = \gamma_{1i_1} + \dots + \gamma_{mi_m}$, where $T = \begin{pmatrix} 1 \dots m \\ i_1 \dots i_m \end{pmatrix}$ is an arbitrary permutation of m numbers.

Definition 1. The **order** of the system (1) is the number R, equal to the maximum of the numbers R_T over all permutations T of m numbers.

Definition 2. Integers $s_1, \dots, s_m$, $t_1, \dots, t_m$ form an **admissible set of numbers** for the system (1) (the matrix Γ) if—

satisfy conditions (2):

$$\sum_{i=1}^{m} (s_i + t_i) = R; \tag{3}$$

$$s_i + t_j \geq \gamma_{ij}. \tag{4}$$

The basis for our further considerations will be the following lemma:

Lemma 1. *For any matrix Γ, whose elements are either nonnegative integers or equal to $-\infty$, there exists an admissible set of integers.*

Thus, let the numbers $s_1, \dots, t_m$ form an admissible set of numbers for system (1). Put

$$A'_{ij}(x, D) = \sum_{|\alpha| = s_i + t_j} a_{ij}^{(\alpha)}(x) D^\alpha. \tag{5}$$

(We note that $A'_{ij} \equiv 0$ if $s_i + t_j > \gamma_{ij}$.)

Introduce the notation

$$A(x, D) = (A_{ij}(x, D)), \qquad A'(x, D) = (A'_{ij}(x, D)). \tag{6}$$

Definition 3. The differential operator $A'(x, D)$, specified by (3), (6), is called the **principal part** of the differential operator $A(x, D)$.

If, in the operators (6), we replace the operators $D_1, \dots, D_n$ by real numbers $\xi_1, \dots, \xi_n$, then we obtain matrices with polynomial coefficients $A(x, \xi)$ and $A'(x, \xi)$. Let

$$\chi(x, \xi) = \det \|A'(x, \xi)\|, \tag{7}$$

$$L(x, \xi) = \det \|A(x, \xi)\|. \tag{8}$$

Definition 4. The polynomial $\chi(x, \xi)$ is called the **characteristic form** of system (1) at the point x.

By virtue of (5), $\chi(x, \xi)$ will be a homogeneous polynomial of degree R in ξ. By virtue of the definition of R, the polynomial $L(x, \xi)$ will contain no terms of degree higher than R. Denote by $l(x, \xi)$ the sum of the terms of degree R of the polynomial $L(x, \xi)$, i.e. the principal part of the determinant of the matrix $A(x, \xi)$.

Lemma 2. *For any values of ξ, the characteristic form $\chi(x, \xi)$ coincides with $l(x, \xi)$.*

We note that Lemma 2 makes it possible to compute the characteristic form of system (1) without constructing the matrix $A'(x, D)$.

Definition 5. System (1) is called **nondegenerate** in the domain G if, for every point $x \in G$, the characteristic form (7) is not identically equal to zero.

Lemmas 1 and 2 make it possible to give an effective definition of ellipticity equivalent to the definition of Douglis and Nirenberg (^{2}).

Definition 6. A nondegenerate system (1) is called **elliptic** in the domain G if, at every point $x \in D$, the characteristic form $\chi(x, \xi)$ is nonzero for all values $\xi \neq 0$.

II. We shall now consider systems of ordinary equations of the form (1), where $D = d/dx$. In the case of ordinary equations, one may restrict attention to nondegenerate systems. The following theorem may serve as a basis for such a restriction.

Theorem 1. *Let system (1) be defined on the interval $[0, X]$, and let the coefficients $a_{ij}^{(\alpha)}(x)$ be meromorphic functions of x. Then either there exists a system of equations*

$$G_i = \sum_{i=1}^{m} B_{ij}(x, D) u_j = g_i, \tag{1a}$$

nondegenerate at all points of the interval $[0, X]$, with the possible exception of a finite number of points, and equivalent to the system* (1), or the system (1^a) is equivalent to such a system of differential (or, possibly, algebraic) equations in which the number of equations is less than the number of unknown functions.

Proof. In view of the degeneracy of the system, there exist meromorphic functions $\lambda_1(x), \dots, \lambda_{k-1}(x)$ such that the k-th row ($k \leqslant n$) of the matrix $A'(x, \xi)$ is expressed in terms of the preceding rows by the formulas

$$A'_{kj}(x,\xi) = \sum_{r=1}^{k-1} \lambda_r(x) A_{rj}(x,\xi)\xi^{s_k - s_r} \quad (s_1 \leqslant s_2 \leqslant \dots \leqslant s_n).$$

Replace the system (1) by the equivalent system of equations

$$F'_i = F_i + \delta_i^k \sum_{r=1}^{k-1} \lambda_r(x) D^{s_k - s_r} F_r = 0. \tag{1'}$$

It is easy to see that $R' < R$, where R' is the order of the system $(1')$. If the system $(1')$ is nondegenerate or $F'_k \equiv 0$, then the theorem is proved; if the system $(1')$ is degenerate, then we replace it by the system $(1'')$, and so on. The theorem is proved.

Consider on the interval $[0, X]$ a nondegenerate system of ordinary equations of order R. By Lemma 1 the system can be written in the form

$$\sum_{j=1}^{m} a_{ij}(x) D^{s_i + t_j} u_j(x) + \sum_{j=1}^{m} \sum_{r < s_i + t_j} a_{ij}^{(r)}(x) D^r u_j(x) = f_i(x), \tag{9}$$

where the matrix $A' = (a_{ij})$, composed of the coefficients of the leading terms of the system (9), is nondegenerate.

Theorem 2. Let the functions a_{ij}, $a_{ij}^{(r)}$ have continuous derivatives up to order $s - s_i$ $(0 = s_1 \leqslant s_2 \leqslant \dots \leqslant s_m = s)$. Then for any point x_0 of the interval $[0, X]$ there is a $\delta > 0$ such that on the interval $[x_0 - \delta, x_0 + \delta]$ the nondegenerate system (9) is equivalent to a system of equations of the form

$$G_i = D^{r_i} u_i + \sum_{j=1}^{m} \sum_{r < r_j} b_{ij}^{(r)}(x) D^r u_j = g_i, \tag{10}$$

where $r_i \geqslant 0$, $r_1 + r_2 + \dots + r_m = R$; $b_{ij}^{(r)}$ are continuous functions of x on the interval $[0, X]$.

Proof. In view of the nondegeneracy of the matrix $A'(x)$, the smoothness of $a_{ij}(x)$, and the smallness of δ, the unknown functions $u_1, \dots, u_m$ can be numbered in such a way that from the system of algebraic equations

$$\sum_{j=1}^{m} a_{ij}(x) \eta_i = \zeta_i$$

one can successively eliminate the unknowns $\eta_1, \dots, \eta_m$ for $|x - x_0| < \delta$. Replace the system (9) by the equivalent system of equations

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$$F_1^{(1)} = a_{11}^{-1} F_1,$$

$$F_k^{(1)} = F_k - a_{1k} a_{11}^{-1} D^{s_k - s_1} F_1, \quad k = 2, \dots, m. \tag{9”}$$

* The system (1) is equivalent to the system (1^a) if every vector-function $u(x)$ (whose components u_i have continuous derivatives on the interval $[x_1, x_2]$ up to orders γ_{ij}) satisfying, for $x_1 \leqslant x \leqslant x_2$, one of the systems (1), (1^a), is required to satisfy the other as well. The numbers x_1 and x_2 are chosen so that on the interval $[x_1, x_2]$ there are no poles of the coefficients of the systems (1) and (1^a).

In system ($9'$) the term $D^{s_1+t_1} u_1$ enters only into the first equation alone. Similarly, we exclude from all equations except the second the terms $D^r u_2$, where $r \geq s_2 + t_2$. Continuing the elimination, we reduce system (9) to the form (10), where $r_i = s_i + t_i,\ i = 1, \dots, m$.

The theorem is proved.

Let us note that by substitutions of the form

$$u_1 = y_1, \quad Du_1 = y_2, \dots,\ D^{r_1 - 1} u_1 = y_1, \quad u_2 = y_{r_1 + 1}, \dots,\ D^{r_m - 1} u_m = y_R$$

the system of equations (10) is reduced to a normal system of first-order equations

$$Dy_i = \sum_{j=1}^{R} c_{ij} y_j. \tag{11}$$

This reduction makes it possible to construct, in a neighborhood of each point of the interval $[0, X]$, sets of R solutions of system (1) forming a fundamental system. Using the theorem on uniqueness for the Cauchy problem for systems of the form (25), one can “glue together” the fundamental systems of solutions defined in a neighborhood of each point of the interval $[0, X]$.

Thus the following theorem may be proved:

Theorem 3. If the system of ordinary equations (1) is nondegenerate on the interval $[0, X]$ and the coefficients $a_{ij}, a_{ij}^{(r)}$ have continuous derivatives up to order $s_n - s_i + \rho$, where $\rho = \max_{k,l}(t_k - t_l)$, then on the interval $[0, X]$ there exist R solutions of system (1) forming a fundamental system.

It follows from Theorem 3 that the number of arbitrary constants in the general solution of a nondegenerate system is equal to its order. The number of arbitrary constants in the general solution of a degenerate system is determined after reducing the latter to nondegenerate form. We note that the number of arbitrary constants in the general solution of a system of equations with constant coefficients is equal to the degree of the polynomial ([5,6]).

This is explained by the fact that a system of equations with constant coefficients (1) can be reduced to the form (10), and even to triangular form, without changing the determinant of the system (8). In the case of degenerate systems with variable coefficients, the true order of the system (the number of arbitrary constants in the general solution), generally speaking, is not related to the order of the polynomial (8), and it is not difficult to give both examples of systems for which it is greater than the order of $L(x, \xi)$, and examples where it is smaller.

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Note: Figure translations are in progress. See original paper for figures.

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