MATHEMATICS

1960

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MATHEMATICS

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THE LURIE METHOD OF DIFFERENTIAL RENTS FOR DETERMINING AN OPTIMAL TRANSPORTATION PLAN

(Presented by Academician I. G. Petrovsky on 28 XII 1959)

Statement of the problem. It is required to specify an optimal method of transporting a product from suppliers with capacities M_i to consumers with capacities A_k, when the unit cost of transportation from the i-th supplier to the k-th consumer is a_{ik}, where

$$\sum M_i = \sum A_k; \qquad a_{ik} \geq 0, \quad M_i > 0, \quad A_k > 0$$

$$(i = 1, \dots, m;\ k = 1, \dots, n).$$

This note gives an exact description of the algorithm of A. L. Lurie' s method of differential rents ([1]) and proves that it leads to the solution of the problem after a finite number of iterations (and operations), bounded by a constant depending only on the numbers m, n. The algorithm is free of all restrictions—nondegenerate cases.

Notation. Let $[a_{ik}]$ be a rectangular matrix (or table) of order (m, n), in which some elements are circled. M_i and A_k are the prescribed capacities and demands. The equality $\sum M_i = \sum A_k$ is not assumed until this is explicitly stated.

Definition 1 (strikability). A system of circles is *strikable* if the circles can be numbered so that each circle lies in a row or column containing no circles with larger numbers; such a numbering is called *striking.*

Algorithm A1 (striking). *Choose (if there is one) a circle that is the only one in its row or in its column, assign it number 1 and strike it out. From the remaining circles again choose such a circle, assign it number 2, etc.*

If the system of circles is strikable, A1 gives a striking numbering; for a nonstrikable system A1 leads to a submatrix in which every row and column contains at least two circles.

Lemma 1. *For strikability it is necessary and sufficient that in every submatrix the number of circles not exceed $m_1 + n_1 - 1$, where (m_1, n_1) is the order of the submatrix.*

Theorem 1. *Let a matrix f with a strikable system of circles be partitioned into four submatrices $f_1 \div f_4$ (see Fig. 1). Suppose there are no circles in the* *submatrices f_1 and f_4. Then, after adding one circle to the submatrix f_4, the system remains strikable.*

f_1	f_2
f_3	f_4

Fig. 1

Definition 2 (distribution). A matrix $[x_{ik}]$ is called a *distribution*, and its coefficients *deliveries* (of capacities $\{M_i\}$ to demands $[A_k]$), if

$$x_{ik} \geq 0; \qquad \mu_i \equiv M_i - \sum_k x_{ik} \geq 0; \qquad \alpha_k \equiv A_k - \sum_i x_{ik} \geq 0. \tag{1}$$

The numbers μ_i and α_k are called **remainders of supplies and capacities**, and $\sum\sum x_{ik}$ is called the **total delivery**.

The numbers $[x_{ik}]$ are called **deliveries according to the system of circles** if (1) is satisfied and $x_{ik} \neq 0$ only if there is a circle in cell i, k.

By $i(j)$ and $k(j)$ we denote the row and the column in which the circle with number j is located. Put $x^j \equiv x_{i(j),k(j)}$. The remainders of supplies and capacities after the first j deliveries have been made will be denoted by μ_i^j and α_k^j.

Algorithm A2 (distribution in order). *Let supplies, capacities, and a numbered system of circles be given. Put*

$$\mu_i^0 = M_i, \qquad \alpha_k^0 = A_k; \qquad x^j = \min(\mu_{i(j)}^{j-1}, \alpha_{k(j)}^{j-1}) \quad (j = 1, 2, ...).$$

Theorem 2. *Algorithm A2, when applied to a crossed-out numbering, gives the maximum total delivery (for the given system of circles).*

Theorem 3. *Suppose that, for a given system of circles and values of supplies and capacities, the supply of one fixed supplier is increased by $\Delta > 0$. Then either, for every $\Delta > 0$, the maximum total delivery (according to the system of circles) from this supplier increases, or, for every $\Delta > 0$, it remains unchanged.*

Owing to this one may introduce

Definition 3 (characteristic of suppliers). For a given system of circles, a supplier and its row are called **surplus** if, when the supply of this one supplier is increased, the maximum total delivery does not increase. Otherwise the supplier and its row are called **deficient**. Circles located in surplus rows (deficient rows) are called **surplus** (respectively **deficient**).

Lemma 2. *For a given numbering of the circles and two sets of supplies*

$$\overline{M}_i \quad \text{and} \quad \widetilde{M}_i = \overline{M}_i + \delta e_i^{i'},$$

where $e_i^{i'} = 1(0)$ *for* $i = i'$ $(i \neq i')$ $(i = 1, 2, ...)$ *(*i' *fixed), the distribution according to A2 has been carried out. Let* $Y \equiv \{j_1, j_2, ...\}$ *be the circles* j *for which* $\tilde{x}^j \neq \overline{x}^j$*, arranged in increasing order of their numbers. Then there exists a* $\Delta > 0$ *such that, for every* $0 < \delta \leq \Delta$*, the following conditions hold: 1) the circles* Y *are the same for all* δ*; 2)* $\tilde{x}^j = \overline{x}^j \pm \delta$ *for* $j = j_x$*, with odd (even)* x*; 3)* $i(j_1) = i'$ *if* Y *is nonempty; 4) the circles* j_x *and* j_{x+1} *are in a common column (row) for odd (even)* x*; 5) the circles* j_σ *and* j_x *with* $\sigma > x + 1$ *cannot be in a common row or column; 6) if row* i' *is deficient (surplus), then all rows* $i(j_1), i(j_2), ...$ *are deficient (surplus) and the number of circles in* Y *is odd (even).*

Theorem 4. *Let* Ξ *be a crossed-out system of circles, and let* σ *be a surplus circle and* ξ *a deficient circle of* Ξ *lying in one column. Then the maximum total delivery and the character (surplus or deficiency) of all suppliers for* Ξ *and for* $\Xi - \xi$ *coincide.*

Theorem 5. *Let* Ξ *be a crossed-out system of circles supplemented by a circle* σ *which is located: 1) in a surplus row of the system* Ξ*; 2) in a column containing no surplus circles of* Ξ*. Suppose that the maximum total deliveries for* Ξ *and* $\Xi + \sigma$ *coincide. Then in* $\Xi + \sigma$*: 1) rows deficient in* Ξ *are deficient; 2) the row of the circle* σ *is deficient.*

Static criteria of the character of suppliers. Let the distribution A2 have been carried out according to a crossed-out numbering of the circles and

$$\sum M_i = \sum A_k.$$

Then: 1) if $\mu_i > 0$, then row i is surplus; 2) if $\mu_k > 0$, then the circles of column k are deficient; 3) in the column of a surplus circle, all circles with positive deliveries are surplus.

Theorem 6. *Criterion 2), with the subsequent repeated application of criterion 3), gives all deficient suppliers.*

Let us note that an analogous application of criteria 1) and 3) can also give all surplus suppliers.

The **method of differential rents** consists in repeating a process, one cycle of which we shall break down into steps and describe. Here, as in what follows,

$$\sum M_i = \sum A_k.$$

1′. The initial system of circles is formed in the first cycle. The price matrix $[a_{ik}^1] = [a_{ik}]$ is introduced, and in each column one of the minimum elements is circled. The resulting system of circles is denoted by s^1.

Before the beginning of the next cycle $p > 1$ there are: the system of circles s^{p-1}, the characters of the suppliers with respect to s^{p-1}, and the price matrix a_{ik}^p, obtained in cycle $p - 1$. The next cycle $p > 1$ begins with step 1^0.

1^0. Formation of a new system of circles s^p. 1) Find an element $a^p_{i'k'}$, minimal in its column, lying in a surplus row and in a column containing no surplus circles. 2) Remove from s^{p-1} the deficient circles located in columns containing surplus circles. Adding to the remaining circles the circle (i', k'), we obtain the system s^p. If in cycle $p-1$ not all the product was distributed, then the desired element $a^p_{i'k'}$ exists, and the system s^p is cancellable.

For all $p = 1, 2, ...$, the following steps are carried out in the same way.

2^0. The **cancelling numbering** of the circles s_p is obtained by means of A1.

3^0. Distribution is carried out according to A2 using the cancelling numbering obtained in 2^0. If all the product is distributed, then the process is finished. Otherwise we proceed to 4^0.

4^0. Determination of the character (surplus or deficiency) of the suppliers may be performed according to definition 3 by means of 3^0. In this, for each supplier $O(m+n)$ operations will be required. For some suppliers it is more advantageous to use static criteria. One may use F. Filler' s criterion: if the new circle of the system s^p is surplus (deficient), then the surplus (deficient) rows with respect to the system s^{p-1} will also be surplus (deficient) with respect to s^p.

5^0. Lifting of rents. 1) Put

$$\Delta^p = \min_{k'} \left(\min_{i'} a^p_{i'k} - \min_{i} a^p_{ik} \right),$$

where the minimum with respect to k' is taken over columns k having no surplus circles, and the minimum with respect to i' is taken over surplus rows i (of the system s^p).

2) To all elements a^p_{ik} standing in deficient rows, Δ^p is added; the remaining elements are not changed. The obtained elements are denoted by a^{p+1}_{ik}. If in cycle p not all the product was distributed, then the minima in the formula for Δ^p are taken over nonempty sets, and therefore the number $\Delta^p \geq 0$ is defined.

Next we pass to 1^0 of cycle $p+1$.

If $\sum M_i = \sum A_k$ and before cycle p not all the product has been distributed, then it is proved successively that: 1) the price matrix $a^{(p)}_{ik}$ and the system of circles s^p are defined; the system s^p is cancellable and, for $p > 1$, contains a circle not belonging to s^{p-1}. 2) In passing from cycle $p-1$ to cycle $p > 1$, the total delivery of the distribution 3^0 does not decrease; if it is preserved, then the number of deficient suppliers increases. 3) The algorithm of the method of differential rents leads to the distribution of the entire product in a finite number of cycles, bounded by a constant $k(m, n)$ depending only on the number of suppliers and consumers. 4) The circles of the system s_p

fall only on the minimal elements of the columns $[a_{ik}^p]$. 5) The complete distribution delivered by the method of differential rents minimizes transportation costs at the prices $[a_{ik}^l]$.

Theorem 7. *The method of differential rents leads to the distribution of the entire product in a finite number of cycles, bounded by a constant depending only on the number of suppliers and consumers. The distribution thereby obtained minimizes the total transportation costs. At the next iteration the system of deliveries remains crossed out, and the quantity of the distributed product does not decrease; if it remains the same, then the number of deficient suppliers increases.*

Example. Let us find the optimal distribution of the product for the transportation prices indicated in Table 1. In the solution the following notation is used: Δ is the increment, s is the total delivery; the deliveries x_{ik} are arranged in the order in which the circles are crossed out; the signs + (−) mark surplus (respectively deficient) rows; the numbers on which the circles fall are set in semibold type.

Table 1

M_i	A_k	1	1	2	1	1	2	1	1	2	1	1	2	1	1	2
		1	2	3	4	5										
1		**3**	1	2+	**3**	1	2−	**3**	1	2+	**3**	1	2−	**4**	2	3
2		6	1	3+	6	1	3+	6	1	3+	6	1	3+	6	1	3
1		4	0	1−	5	1	2−	5	1	2−	5	1	2−	6	2	3
$\Delta =$					1			0			0			1		
$s =$			2			2			3			3			4	
		$x_{11} = 1$	$x_{11} = 1$	$x_{11} = 1$	$x_{22} = 1$	$x_{33} = 1$										
		$x_{32} = 1$	$x_{33} = 1$	$x_{33} = 1$	$x_{33} = 1$	$x_{11} = 1$										
		$x_{33} = 0$	$x_{12} = 0$	$x_{12} = 0$	$x_{11} = 1$	$x_{22} = 1$										
			$x_{32} = 0$	$x_{22} = 1$	$x_{12} = 0$	$x_{13} = 0$										
				$x_{32} = 0$	$x_{13} = 0$	$x_{23} = 1$										

The solution of the example is unique; the transportation costs are equal to 8.

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Received
21 XII 1959

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