MATHEMATICS

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Abstract

Full Text

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ON DOMAINS OF UNIVALENCE OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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If the derivative $P_n'(z)$ of a polynomial $P_n(z)$ of degree $n > 1$ does not vanish in the disk $|z| < 1$, then, by S. Kakeya' s theorem (^{1}), the polynomial $P_n(z)$ is univalent in the concentric disk of radius $\sin \frac{\pi}{n}$. We shall show how the radius of univalence in Kakeya' s theorem can be increased by taking into account the distribution of the zeros of $P_n'(z)$ outside the unit disk. The method we use is based on the following lemma.

Lemma. Let $n > 1$ be an integer. Consider n positive numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, whose sum $S = \sum \alpha_k$ does not exceed $\pi/2$, and n complex numbers $u_1, u_2, \dots, u_n$, whose moduli satisfy the inequalities

$$|u_k| \leq \sin \alpha_k, \qquad k = 1, 2, \dots, n. \tag{1}$$

Under these assumptions, the real part of the product $\prod_k (1+u_k)$ is nonnegative:

$$\operatorname{Re} \prod_k (1 + u_k) \geq 0. \tag{2}$$

The equality sign in relation (2) occurs only when $S = \pi/2$, $u_k = i \sin \alpha_k e^{i\alpha_k}$, $k = 1, 2, \dots, n$, or $u_k = -i \sin \alpha_k e^{i\alpha_k}$, $k = 1, 2, \dots, n$.

Proof. Put $u_k = r_k e^{i\varphi_k}$, $1 + u_k = \rho_k e^{i\psi_k}$, where $0 < r_k \leq \sin \alpha_k$ and $\rho_k > 0$. We may assume that $-\pi/2 < \psi_k < \pi/2$, since $\cos \psi_k > 0$. Eliminating ρ_k from the equations

$$1 + r_k \cos \varphi_k = \rho_k \cos \psi_k, \qquad r_k \sin \varphi_k = \rho_k \sin \psi_k,$$

we find

$$\sin \psi_k = r_k \sin(\varphi_k - \psi_k),$$

$$|\sin\psi_k| \le r_k \le \sin\alpha_k, \qquad |\psi_k| \le \alpha_k, \tag{3}$$

$$\left|\sum_k \psi_k\right| \le \sum \alpha_k \le \frac{\pi}{2}.$$

The assertions of the lemma follow from these relations.

Theorem 1. Let $Q(z)$ be a polynomial in z of degree $n > 1$, whose roots $z_1, z_2, \dots, z_n$ are nonzero, and let $|z_k| = r_k$. Then the polynomial of degree $n+1$

$$P(z) = \int Q(z)\,dz$$

univalent in the disk $|z| \leqslant r$, where r denotes the positive root of the equation

$$\sum_k \arcsin\frac{r}{r_k} = \frac{\pi}{2}.$$

Proof. Without loss of generality, we may assume that $Q(0) = 1$ and, consequently,

$$Q(z) = \prod_k \left(1 - \frac{z}{z_k}\right). \tag{4}$$

Choosing arbitrarily n positive numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, subject to the condition $\sum \alpha_k \leqslant \frac{\pi}{2}$, and applying the lemma to the product (4), we conclude that the real part of $Q(z)$ is nonnegative if

$$\frac{|z|}{r_k} \leqslant \sin\alpha_k, \qquad |z| \leqslant r_k \sin\alpha_k,$$

that is, if $|z|$ does not exceed the least of the products

$$r_k \sin\alpha_k, \qquad k = 1, 2, \dots, n.$$

In order to choose the α_k in the most advantageous way, let us note that one can prove the existence of n positive numbers $\alpha'_1, \alpha'_2, \dots, \alpha'_n$, subject to the inequality $\sum \alpha'_k \leqslant \frac{\pi}{2}$, for which the least of the numbers

$$r_1 \sin\alpha'_1,\ r_2 \sin\alpha'_2, \dots,\ r_n \sin\alpha'_n \tag{5}$$

has the greatest possible value. We shall show that for this it is necessary and sufficient that the conditions

$$\sum \alpha'_k = \frac{\pi}{2}, \qquad r_1 \sin \alpha'_1 = r_2 \sin \alpha'_2 = \dots = r_n \sin \alpha'_n$$

be satisfied.

Indeed, let the number μ—the least of the numbers (5)—have the greatest possible value. If $\sum \alpha'_k < \frac{\pi}{2}$, then one can determine a positive number ε so that the inequality $\sum \alpha_k < \frac{\pi}{2}$ holds, where $\alpha_k = \alpha'_k + \varepsilon$. In that case the least of the numbers $r_k \sin \alpha_k$ will be greater than μ, which is impossible. On the other hand, if the numbers (5) are not equal to one another, then among them at least one, for example $r_m \sin \alpha'_m$, is greater than μ. Putting $\alpha_k = \alpha'_k + \varepsilon$ for $k \neq m$ and $\alpha_m = \alpha'_m - (n-1)\varepsilon$, we conclude that $\sum \alpha_k = \sum \alpha'_k$ and that, for sufficiently small positive ε, each of the numbers $r_k \sin \alpha_k$ is greater than μ, i.e., we again arrive at a contradiction.

If we denote by r the common value of the numbers (5) and eliminate $\alpha'_1, \alpha'_2, \dots, \alpha'_n$ from the equations

$$\sum \alpha'_k = \frac{\pi}{2}, \qquad r_k \sin \alpha'_k = r \quad (k = 1, 2, \dots, n),$$

then for r we obtain the equation

$$\sum_k \arcsin \frac{r}{r_k} = \frac{\pi}{2}, \tag{6}$$

which has exactly one positive root, since its left-hand side is an increasing function of r when r varies from 0 to

smallest of the numbers r_k. In the disk $|z| < r$ we have $\operatorname{Re} P'(z) > 0$, whence, as is known, it follows that the function $P(z)$ is univalent in $|z| \leq r$. Thus the theorem is proved.

The following theorem gives more concrete indications concerning the value of the radius r, depending on the distribution of the roots of $P'(z) = Q(z)$.

Theorem 2. Denote by m a natural number $< \dfrac{n+1}{2}$ and by R the number

$$R = \sin \frac{\pi}{n+1} : \sin \frac{(n+1-2m)\pi}{(n-m)(2n+2)}.$$

If m of the roots of the polynomial

$$Q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right)$$

lie in the annulus $1 \leq |z| \leq R$, and the remaining $n-m$ lie in the region $|z| > R$, then the radius of univalence r of the polynomial

$$P(z) = \int Q(z)\,dz,$$

defined from equation (6), will be greater than the radius

$$\sin\frac{\pi}{n+1},$$

which is given by Kakeya' s theorem.

Proof. Let r be the positive root of equation (6), where $r_k = |z_k|$ and $0 < r_1 \leq r_2 \leq \dots \leq r_n$. Write equation (6) in the form

$$\sum_1^m \arcsin\frac{r}{r_k} + \sum_{m+1}^n \arcsin\frac{r}{r_k} = \frac{\pi}{2}.$$

In the first sum we substitute 1 for r_k, and in the second, R. Then we obtain the inequality

$$m\arcsin r + (n-m)\arcsin\frac{r}{R} > \frac{\pi}{2},$$

from which it follows that the positive root r' of the equation

$$m\arcsin r' + (n-m)\arcsin\frac{r'}{R} = \frac{\pi}{2}$$

is smaller than r. Substituting the value of R, we easily find

$$r' = \sin\frac{\pi}{n+1}.$$

Thus $r > \sin\frac{\pi}{n+1}$, which proves the theorem.

Corollary. For $m = 1$ we have

$$R = 2\cos\frac{\pi}{2n+2}.$$

From Theorem 2 we conclude that if $|z_1| = 1$ and $|z_k| > 2\cos\frac{\pi}{2n+2}$ for $k = 2, 3, \dots, n$, then the polynomial

$$P(z) = \int Q(z)\,dz$$

is univalent in some disk $|z| \leq r$, where

$$r > \sin\frac{\pi}{n+1}.$$

In conclusion we note that if the polynomials $Q_1(z)$ and $Q_2(z)$, of degrees p and q, do not vanish at $z = 0$, then our method can be applied to the function

$$P(z) = \int \frac{Q_1(z)}{Q_2(z)}\, dz,$$

i.e. one can establish its univalence in some disk with center $z = 0$. If

$$Q_1(z) = \prod_{k=1}^{p} \left(1 - \frac{z}{z_k}\right)$$

and

$$Q_2(z) = \prod_{k=p+1}^{p+q} \left(1 - \frac{z}{z_k}\right),$$

then it is easy to prove that in the disk $|z| \le r$, where r is the positive root of the equation

$$\sum_{1}^{p+q} \arcsin \frac{r}{|z_k|} = \frac{\pi}{2},$$

the real part of the function

$$R(r) = \frac{Q_1(z)}{Q_2(z)}$$

is nonnegative, and the function $P(z)$ is univalent.

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References

1. S. Kakeya, Tôhoku Math. J., **11**, 5 (1917).

Note: Figure translations are in progress. See original paper for figures.

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