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Abstract

Full Text

Mathematical Physics

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On the Uniqueness of the Solution of Integral Equations of Antenna Theory (of the First Kind)

(Presented by Academician N. N. Bogolyubov, 3 I 1960)

In paper ⁽¹⁾, integral equations (11) and (12) of Fredholm type of the first kind were obtained for the density of electric current in an antenna, which is a body of revolution with an impedance surface. In the present paper the question of the uniqueness of the solution of the indicated integral equations is considered for the case of a perfectly conducting antenna.

We use the following notation: S is the surface of the antenna; G is the region inside it; D is the region outside S ; the coordinate system is cylindrical (z, ρ, φ) ; the remaining notation is the same as in ⁽¹⁾, except that everywhere $\zeta_1 = \zeta_2 = 0$.

It is assumed that the surface S and the functions considered on the surface (E_t and j) satisfy all smoothness conditions necessary for the existence and uniqueness of the solution of the exterior boundary-value problem of electrodynamics (see, for example, ^(2, 3)).

1. Equation for the Density of the Meridional Current (equation (12) of paper ⁽¹⁾)

$$\int_L \left[\cos \theta' \frac{\partial^2 \psi}{\partial \rho' \partial z} + \sin \theta' \left(\frac{\partial^2 \psi}{\partial z^2} + k^2 \psi \right) \right] j_1^{(0)} \rho' dl' + i\omega \varepsilon \int_L \frac{\partial \psi}{\partial \rho'} E'_\tau \rho' dl' = 0. \quad (1)$$

This equation corresponds to the case of an axisymmetric field with longitudinal excitation of electric type $\mathbf{H} = \{0, 0, H_\varphi(\rho, z)\}$, and is obtained in ⁽¹⁾ from the requirement that, in the region G , the derivative with respect to ρ of the magnetic field vector be equal to zero on the z -axis,

$$\left. \frac{\partial H_\varphi}{\partial \rho} \right|_{\rho=0} = 0, \quad (1)$$

where the field is expressed by the Stratton-Chu formula for the exterior problem.

Moreover, from Maxwell's equations it follows that H_φ satisfies the equation

$$\frac{\partial^2 H_\varphi}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial H_\varphi}{\partial \rho} \right) - \left(\frac{1}{\rho^2} - k^2 \right) H_\varphi = 0. \quad (2)$$

Considering the behavior of the analytic in G solution of equation (2) as $\rho \rightarrow 0$, we obtain that the general solution of this equation has the form

$$H_\varphi = \text{const} \cdot \rho H(\rho, z), \quad H(0, z) \neq 0, \quad (3)$$

whence it follows that equation (2) has only the trivial solution $H_\varphi \equiv 0$ satisfying condition (1). In other words, the following general assertion holds:

An axisymmetric electromagnetic field of electric type

$$\mathbf{H} = \{0, 0, H_\varphi(\rho, z)\},$$

whose derivative with respect to ρ of the magnetic field vector is equal to zero on the axis of symmetry, is identically equal to zero everywhere in the domain of analyticity of the given field.

It follows from this that the electromagnetic field given for the exterior problem by the Stratton-Chu formulas, with the prescribed excitation E'_{τ_0} substituted into them and with the value j_{τ_0} found from equation (I), will be identically equal to zero everywhere in the domain G , i.e. $H(M) \equiv 0$, $E(M) \equiv 0$ for $M \in G$, where

$$\begin{aligned} \mathbf{H}(M) &= -\frac{1}{4\pi} \int_S \{i\omega\varepsilon[\mathbf{n}(N)\mathbf{E}(N)]\psi(M, N) - [\mathbf{j}_{\tau_0} \text{grad}' \psi]\} dS, \\ \mathbf{E}(M) &= \frac{1}{4\pi} \int_S \{i\omega\mu \mathbf{j}_{\tau_0} \psi(M, N) + [[\mathbf{n}(N)\mathbf{E}(N)] \text{grad}' \psi] + \\ &\quad + (\mathbf{n}(N)\mathbf{E}(N)) \text{grad}' \psi\} dS, \end{aligned} \quad (4)$$

$$[\mathbf{n}(N)\mathbf{E}(N)] = -E'_{\tau_0}[\mathbf{n}(N)\vec{\tau}(N)], \quad (\mathbf{n}(N)\mathbf{E}(N)) = \frac{1}{i\omega\varepsilon} \left(\frac{\cos \theta'}{\rho'} j_{\tau_0} - \frac{\partial j_{\tau_0}}{\partial l'} \right).$$

Considering the Maxwell field given by formulas (4) in the domain D , and letting the point M from D tend to the surface S , as well as letting the point M tend to S in formulas (4) for $M \in G$, we obtain, for the tangential components of the field vectors on the surface S , the values

$$[\mathbf{nH}]|_S = \mathbf{j}_{\tau_0}, \quad [\mathbf{nE}]|_S = -E'_{\tau_0}[\mathbf{n}\vec{\tau}],$$

i.e. the tangential component of the electric vector assumes on the surface the prescribed value.

Further, since the radiation conditions are satisfied automatically, the field defined by the Stratton-Chu formulas will also be a solution of the exterior boundary-value problem of electrodynamics. From the uniqueness of its solution follows the uniqueness of the solution of equation (I).

2. Equation for the density of the azimuthal current (the first of equations (11) in paper ⁽¹⁾).

$$\int_L \frac{\partial \psi}{\partial \rho'} j_2^{(0)} \rho' dl' = \int_L \left[\left(i\omega \varepsilon \psi \sin \theta' + \frac{\cos \theta'}{i\omega \mu \rho'} \frac{\partial \psi}{\partial z} \right) E'_{\varphi 0} - \frac{1}{i\omega \mu} \frac{\partial \psi}{\partial z} \frac{\partial E'_{\varphi 0}}{\partial l'} \right] \rho' dl'. \quad (\text{II})$$

This equation corresponds to the case of axisymmetric excitation of magnetic type $\mathbf{E} = \{0, 0, E_\varphi\}$ and was obtained in ⁽¹⁾ from the requirement that the z -component of the magnetic field vector, given by the Stratton-Chu formula for the exterior problem, be equal to zero on the z -axis in the domain G . From Maxwell's equation

$$\frac{E_\varphi}{\rho} + \frac{\partial E_\varphi}{\partial \rho} = -i\omega \mu H_z$$

and the condition $H_z|_{\rho=0} = 0$ we find that

$$\left. \frac{\partial E_\varphi}{\partial \rho} \right|_{\rho=0} = 0.$$

Moreover, E_φ also satisfies equation (2). Further, as in item 1, the following assertion follows:

An axisymmetric field of magnetic type $\mathbf{E} = \{0, 0, E_\varphi\}$, the z -component of whose magnetic vector is equal to zero on the axis of symmetry, will be equal to zero everywhere in the domain of analyticity of the given field.

As in the preceding item, from this follows the uniqueness of the solution of the integral equation of the first kind for the azimuthal component of the current density.

3. Equations of the first kind for the exterior scalar wave problem and equations for the first harmonics of the current density.

- a) In finding the solution of the exterior boundary-value problem for the equation $\Delta u + k^2 u = 0$, integral equations of the second and first kind, analogous to the equations in ⁽¹⁾ for the density of the electric current, may be applied. Here only equations of the first kind are considered.

As the initial equation for finding $\partial u/\partial n$ on S (the first problem) or u (the second problem), one uses Green's scalar formula, written for the exterior problem:

$$u(M) = \frac{1}{4\pi} \int_S \left(\frac{\partial u}{\partial n_N} \psi(M, N) - u \frac{\partial \psi}{\partial n_N} \right) ds = 0, \quad M \in G. \quad (5)$$

Here $\psi(M, N) = e^{ikr}/r$, $r = \sqrt{\rho^2 + \rho o'^2 + 2\rho\rho' \cos(\varphi - \varphi') + (z - z')^2}$.

From the uniqueness of the solution of the exterior problem and from the properties of simple- and double-layer potentials (see (2)), it follows that equation (5), both for the first and for the second problem, has a unique solution. Using the analyticity of (5) in z and in ρ inside the domain G , we require (5) to be satisfied not in the whole domain G , but only in some subdomain of it with nonzero volume. Then (5) will again be satisfied in the whole domain G , i.e., the uniqueness of the solution of the equation is preserved. Further, requiring (5) to hold only for points M lying on the axis z , we obtain the equation

$$\int_L \frac{\partial u}{\partial n_N} \psi(M, N) \Big|_{\rho=0} \rho' dl' = \int_L u \frac{\partial \psi}{\partial n_N} \Big|_{\rho=0} \rho' dl', \quad (6)$$

which, generally speaking, will no longer have a unique solution. And only in the axisymmetric case, using the equation $\Delta u + k^2 u = 0$, can one show (analogously to how this was done in item 1) that, for both the first and the second boundary-value problems, equation (6) will have a unique solution. In other words, the kernels of the equations

$$\psi(M, N) \Big|_{\rho=0} = \psi(l, l') = \frac{e^{ikr}}{r} \rho'(l'), \quad (7)$$

$$\frac{\partial \psi}{\partial n_N} \Big|_{\rho=0} = \frac{\partial \psi}{\partial n_N}(l, l') = \left(\cos \theta' \frac{\partial \psi}{\partial z'} + \sin \theta' \frac{\partial \psi}{\partial \rho'} \right) \rho'(l') \quad (8)$$

will be closed (for the notation see (11) in (1)).

- b) It is not possible to carry out the investigation of the uniqueness of the solution for the second and third equations in (11) from (1), using the analytic properties of Maxwell fields.

However, it is not difficult to see that, for an antenna with an infinitely conducting surface ($\zeta_1 = \zeta_2 = 0$), the kernel of the equation for the first harmonic of the meridional-current density (the third equation in (11) from (1))

$$\int_L \left(\cos \theta' \frac{\partial \psi}{\partial z'} + \sin \theta' \frac{\partial \psi}{\partial \rho'} \right) j_{1m}^{(1)} \rho' dl' =$$

$$= \int_L \left\{ i\omega\varepsilon\psi E'_{\tau m} - \frac{1}{i\omega\mu} \frac{\partial\psi}{\partial\rho'} \left[\frac{E'_{\tau m}}{\rho'} + \lambda \left(\frac{\partial E'_{\varphi n}}{\partial l'} - \frac{\cos\theta'}{\rho'} E'_{\varphi n} \right) \right] \right\} \rho' dl' \quad (\text{III})$$

$$(m = s, n = c, \lambda = -1 \text{ or } m = c, n = s, \lambda = 1)$$

will be identical to the kernel of the equation for the axisymmetric scalar problem (8), whose closedness was proved above. Hence follows the uniqueness of the solution of equation (III).

The second equation in (11) (for the first harmonic of the azimuthal component of the current density) in the case $\zeta_2 = 0$

$$\int_L \frac{\partial\psi}{\partial z} j_{2m}^{(1)} \rho' dl' = \int_L i\omega\varepsilon\psi \cos\theta' E'_{\varphi m} \rho' dl' \quad (m = s \text{ or } m = c) \quad (\text{IV})$$

will have a nonunique solution of the form $\nu = \nu_1 + c\nu_0$, where c is an arbitrary constant, ν_1 is some particular solution of the equation, and ν_0 is ...

the solution of the equation

$$\int_L \psi(l, l') v_0(l') \rho' dl' = 1; \quad (9)$$

for $\psi(l, l')$, see formula (7). We shall not consider the question of the existence of a solution of equation (9); we note only that for surfaces S and values of k for which the equation

$$\frac{1}{2\pi} \int_S \frac{\partial\psi(M, N)}{\partial n_N} v(N) dS - v(M) = 0, \quad M, N \in S,$$

has no eigenvalues, a solution of (9) exists. It is also obvious that a solution exists for an infinite cylinder (the solution is constant). Therefore, to find the first harmonic of the azimuthal component of the current density, it is proposed to use the equation

$$\begin{aligned} & \int_L \left\{ \left[k^2\psi + \frac{1}{\rho'} \frac{\partial\psi}{\partial\rho'} + \zeta_2 \left(\cos\theta' \frac{\partial\psi}{\partial z'} + \sin\theta' \frac{\partial\psi}{\partial\rho'} \right) i\omega\varepsilon \right] j_{2m}^{(1)} \right. \\ & \quad \left. - \lambda \frac{\partial\psi}{\partial\rho'} \left(\frac{\partial j_{1m}^{(1)}}{\partial l'} - \frac{\cos\theta'}{\rho'} j_{1n}^{(1)} \right) \right\} \rho' dl' \\ & = i\omega\varepsilon \int_L \left(\cos\theta' \frac{\partial\psi}{\partial z'} + \sin\theta' \frac{\partial\psi}{\partial\rho'} \right) E_{\varphi m} \rho' dl' \end{aligned} \quad (10)$$

$$(m = s, n = c, \lambda = -1 \text{ or } m = c, n = s, \lambda = 1),$$

obtained in the same way as equations (I), (II), (III), (IV) in (1), but proceeding from the requirement $\mathbf{E}|_{\rho=0} = 0$.

The question of the existence of solutions in equations (I), (II), (III), (IV) in (1) and in (6), (10) is not discussed, since the existence of a solution for the exterior boundary-value problem of electrodynamics and for the scalar wave equation implies the existence of solutions in the equations under consideration.

Let us note that the results obtained in §§ 1 and 2 can also be generalized to the case of the problem with the Leontovich boundary condition ($\zeta_1 \neq 0, \zeta_2 \neq 0$).

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Note: Figure translations are in progress. See original paper for figures.

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