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Abstract

Full Text

MATHEMATICS

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ON THE DIFFERENTIATION OF COMPLEX FUNCTIONS

(Presented by Academician A. N. Kolmogorov, 10 IX 1959)

§ 1. Let E be some set in the complex plane; ζ a limit point of this set, $\zeta \in E$; $f(z)$ a complex function (which may also take infinite values) defined on E.

A number a (finite or not) is called a **derivative number** of the function $f(z)$ at the point ζ (with respect to the set E) if there exists a sequence of complex numbers $\{z_n\}$ such that: 1) $z_n \in E$ $(n = 1, 2, ...)$, $z_n \to \zeta$ as $n \to \infty$; 2) all the differences $f(z_n) - f(\zeta)$ have meaning (i.e., for any n at least one of the numbers $f(z_n)$, $f(\zeta)$ is different from infinity); 3)

$$a = \lim_{n\to\infty} \frac{f(z_n) - f(\zeta)}{z_n - \zeta},$$

if $\zeta \neq \infty$, or

$$a = \lim_{n\to\infty} z_n[f(z_n) - f(\zeta)],$$

if $\zeta = \infty$.

Let $\mathfrak{M}_f(\zeta)$ be the set of all derivative numbers of the function $f(z)$ at the point $\zeta \in E$ with respect to E.

Theorem 1. *For an arbitrary complex function $f(z)$ (finite or not), defined on some set $*$ E of the complex plane, at almost every point $**$ $\zeta \in E$ one of the following four cases holds:*

1) *$\mathfrak{M}_f(\zeta)$ is some circle $|z - a(\zeta)| = r(\zeta)$, $0 \le r(\zeta) < \infty$ (in particular, when $r(\zeta) = 0$, $\mathfrak{M}_f(\zeta)$ consists of a single point, i.e. in this case $f(z)$ has a derivative at the point ζ);*
2) *$\mathfrak{M}_f(\zeta)$ coincides with the (extended) complex plane;*
3) *$\mathfrak{M}_f(\zeta)$ consists of some circle $|z - a(\zeta)| = r(\zeta)$, $0 \le r(\zeta) < \infty$, and the point $z = \infty$;*
4) *$\mathfrak{M}_f(\zeta)$ consists of the single point $z = \infty$.*

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Remark 1. At almost every point of finiteness of $f(z)$, one of the first three cases holds.

Remark 2. At almost every point of discontinuity of $f(z)$, one of the first two cases holds.

Remark 3. At points $\zeta \in E$ at which the real and imaginary parts of $f(z)$ have total differentials (with respect to the set E), case 1) holds; conversely, almost everywhere where case 1) holds, the real and imaginary parts of $f(z)$ have a total differential (with respect to E).

Remark 4. Each of the four cases may hold on a set of positive measure (concerning the occurrence of the first two of them, see ([2,3])).

The theorem just formulated is an analogue of the well-known theorem of Denjoy ([1]) on derivative numbers of a real function of a real variable. It generalizes a result of Yu. Yu. Trokhimchuk ([2]), obtained for functions continuous in a domain.

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* Generally speaking, nonmeasurable.

** That is, with the possible exception of points $\zeta \in E$ which together form a set of measure zero.

§ 2. Let, as before, E be some set in the complex plane, but $\infty \notin E$. Let, further, $f(z)$ be a complex function, defined and finite on E, and let $\zeta \in E$ be a limit point of E. We say that $f(z)$ has k Vallee-Poussin derivatives at the point ζ (with respect to the set E) if, for $\zeta + h \in E$,

$$f(z+h) = f(z) + hf_{(1)}(z) + \frac{h^2}{2!}f_{(2)}(z) + \cdots + \frac{h^k}{k}f_{(k)}(z) + o(h^k).$$

Here the number $f_{(i)}(z)$ $(1 \leq i \leq k)$ (determined, obviously, uniquely) is called the i-th Vallee-Poussin derivative (the i-th $(V-P)$-derivative).

The advantage of $(V-P)$-differentiation over ordinary, successive differentiation consists in the fact that the k-th $(V-P)$-derivative is defined "at a point," independently of whether or not the preceding $(V-P)$-derivatives exist in some neighborhood of this point. Note that the first $(V-P)$-derivative coincides with the ordinary first derivative, and that from the existence on the set $e \subset E$ of k $(V-P)$-derivatives it follows that the function $f(z)$ is approximately differentiable k times almost everywhere on e.

Below $R_n[f]$ denotes the best approximation to the function $f(z)$ by rational functions of order not exceeding n* (i.e. $R_n[f] = \inf_\varphi\{\sup_{z\in E}|f(z) - \varphi(z)|\}$, where φ ranges over all rational functions of order not exceeding n); $R_n(z)$ denotes a rational function of order not exceeding n which realizes this approximation ($R_n(z)$ exists [4]).

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Theorem 2. *If, for a function $f(z)$ defined on some set E,*

$$R_n[f] \leq \frac{C}{n^{p+\varepsilon}},$$

where $C < \infty$ does not depend on n, p is natural, and $\varepsilon > 0$, then almost everywhere—with respect to planar measure if E is planar, and with respect to linear measure if E belongs to the real axis—$f(z)$ has p Vallee-Poussin derivatives, and

$$f_{(i)}(z) = \lim_{k\to\infty} R'_{n_k}(z) \qquad (1 \leq i \leq p)$$

almost everywhere (with respect to the corresponding measure) on E, whatever lacunary sequence $\{n_k\}$ may be.

In the case where $E = [0,1]$, this theorem substantially strengthens Theorem 3 of A. A. Gonchar' s paper [(5)], and also contains Theorem 2 of that paper.

Theorem 3. *Let a sequence of nonnegative numbers $\{a_n\}$ be nonincreasing, and let E be a set of points on the real axis. In order that every function $f(x)$ $(x \in E)$ for which $R_n[f] \leq a_n$ be differentiable almost everywhere on E (with respect to E), it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} a_n < \infty. \tag{1}$$

In general, more can be proved, namely:

If $R_n[f] \leq a_n$ and (1) is satisfied, then $f(x)$ coincides on E with some function $f^*(x)$, absolutely continuous on the entire line $(-\infty,\infty)$, and almost everywhere on E

$$f'(x) = \lim_{k\to\infty} R'_{n_k}(x)$$

*for any lacunary sequence of numbers n_k**.*

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* The order of a rational function $\varphi(z) = \dfrac{a_m z^m + \cdots + a_0}{b_k z^k + \cdots + b_0}$ is called $n = \max\{m,k\}$.

** It is possible that the condition $\sum_{n=1}^{\infty} a_n < \infty$ is not only sufficient for the absolute continuity of $f(z)$, but also necessary.

If, however,

$$\sum_{n=1}^{\infty} a_n = \infty,$$

then on E there exists a function $f_1(x)$, nowhere on E (with respect to E) differentiable, and a function $f_2(x)$, almost nowhere on E approximately differentiable, such that $R_n[f_i] \leq a_n$ $(i = 1, 2)$. This fact is of fundamental importance, since it says that the minimal conditions on $R_n[f]$ ensuring approximate differentiability (almost everywhere) coincide with the minimal conditions on $R_n[f]$ ensuring ordinary differentiability (also almost everywhere), despite the fact that the requirement of approximate differentiability is much weaker than the requirement of ordinary differentiability.

§ 3. Let K be a continuum (i.e., a bounded closed connected set) lying in the complex plane Z (K may be nowhere dense in Z). The complement of K decomposes into a finite or countable set of nonintersecting simply connected domains g_i with boundaries, respectively, γ_i. We shall denote the diameter of γ_i by $d(\gamma_i)$. Consider the set consisting of all functions that have a continuous derivative on K, and all uniform (on K) limits of these functions, and denote it by $A[K]$. The space $A[K]$ is a Banach space* and, by a theorem of A. G. Vitushkin[6], coincides with the set of all functions admitting uniform (on K) approximation with arbitrary accuracy by rational functions.

In [7] it is shown that no conditions on the continuum K ensure ordinary differentiability of functions from $A[K]$ at any point $\xi \in K$. However, the following is true:

Theorem 4. *Let the continuum K be such that*

$$\sum_i [d(\gamma_i)]^{\frac{2}{m+2}} \quad (m \text{ natural}). \tag{2}$$

Then K can be represented in the form

$$K = G \cup \Gamma,$$

where $\Gamma \supset \bigcup_i \gamma_i$, $\operatorname{mes} \Gamma = \operatorname{mes} \bigcup_i \gamma_i$, and G is the limit of an increasing sequence of closed sets E_n $(E_n \subset E_{n+1})$ possessing the following properties:

1) *if $f \in A[K]$, then on each set E_n the function $f(z)$ is m times differentiable with respect to this set;*

2) *if a sequence $\{f_k(z)\}$ of functions from $A[K]$ converges uniformly (on K) to a function $f(z)$, then $f \in A[K]$, and on each set E_n the derivatives of $f_k(z)$ (with respect to the set E_n) up to order m inclusive converge uniformly to the corresponding derivatives of $f(z)$.*

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This theorem is an analogue of Weierstrass' theorem on a sequence of functions analytic in a domain G and converging uniformly inside the domain.

Let us note that if, instead of (2), one requires somewhat more, namely

$$\sum_i [d(\gamma_i)]^{\frac{1}{m+2}} < \infty, \tag{2'}$$

then each set E_n will consist of a finite number of continua. If the series (2′) converges for every $m > 0$, then all functions from $A[K]$

* With norm $\|f\| = \max_{z \in K} |f(z)|$.

infinitely differentiable on each E_n, and from the uniform convergence (on K) of $\{f_k(z)\}$ to $f(z)$ it follows that $f \in A[K]$, and also that, on each E_n, all derivatives of $f_k(z)$ converge uniformly on E_n to the corresponding derivatives of $f(z)$.

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