MOTION OF A CONDUCTING PLASMA UNDER THE ACTION OF A PISTON

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Abstract

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G. A. SKURIDIN and K. P. STANYUKOVICH

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In paper (1) it was shown that the approximate (asymptotic) one-dimensional motion of a plasma is described by the relations (for $t \gg 0$)

$$H = A_0(t) \exp\left[\frac{\frac{x^2}{2} + \frac{\alpha}{B} xT}{4xt + \beta}\right] \left\{\frac{x + \frac{\alpha}{B}T}{4xt + \beta} \cos\omega\left(\frac{B}{\alpha}x + T\right) - \frac{B\omega}{\alpha} \sin\omega\left(\frac{B}{\alpha}x + T\right)\right\}; \tag{1}$$

$$\rho = \sqrt{4xt + \beta}\, \Phi(z); \tag{2}$$

$$P + \frac{H^2}{8\pi} = P_0(t) - \int \rho\,(u_t + uu_x)\, dx. \tag{3}$$

Here the gas velocity is determined from the equation

$$u(x, t) = 2x\, \frac{x + \frac{\alpha}{B}T}{4xt + \beta} - \frac{\alpha}{B}\dot{T}. \tag{4}$$

The quantities entering (1)–(4) have the form: $B = \text{const}$; $\beta = \text{const} < 0$; $\alpha = \omega\sqrt{\chi}$, where ω is the frequency; $T(t)$ and $A_0(t)$ are arbitrary functions of time.

The arbitrary function $T(t)$ can be determined by introducing some condition for u, for example, if the wall moves according to the law $x = \psi(t)$. Then

$$T = \sqrt{4xt + \beta}\left\{\text{const} + \frac{B}{\alpha}\int \left[\frac{2x\psi}{4xt + \beta} - \dot{\psi}\right] \frac{dt}{\sqrt{4xt + \beta}}\right\}. \tag{5}$$

The arbitrary functions $\Phi(z)$, where

$$z = \frac{x}{\sqrt{4xt+\beta}} + \int \left[\frac{2x\psi}{4xt+\beta} - \dot{\psi}\right] \frac{dt}{\sqrt{4xt+\beta}}, \tag{6}$$

and $A_0(t)$ are determined from the boundary conditions.

As an application of the general relations obtained, let us consider the following problem: let a conducting medium (plasma) move in a tube under the action of a piston, with the law of motion of the piston having the form

$$x_\text{p} = \frac{at^2}{2} \equiv \psi(t). \tag{7}$$

Then from (4) we have

$$u_\text{p}(t) = at \equiv \dot{\psi}(t). \tag{8}$$

After some time a shock wave arises in front of the piston. For a sufficiently large value of a, the shock wave arises almost immediately and near the origin of coordinates, since the time of formation of the shock wave t_y and the place of its formation x_y are determined by the well-known formulas ([2])

$$t_y = \frac{2c_0}{(k+1)a}, \qquad x_y = \frac{2c_0^2}{(k+1)a},$$

where c_0 is the speed of sound in the medium.

We shall neglect the region of motion of the medium before the formation of the strong shock wave, since it is small and has no substantial significance.

We proceed to determine the quantities entering into (1)–(4). First of all, let us determine $T(t)$. From (5), taking account of (7), (8), we obtain

$$T(t) = \sqrt{4\chi t+\beta}\cdot\text{const} - \frac{Bat^2}{2a}.$$

Since at $t = 0$, $T = 0$, finally

$$T(t) = -\frac{B}{a}\psi(t) = -\frac{Bat^2}{2a}. \tag{9}$$

In this case

$$u(x,t) = -\frac{2x}{4\chi t+\beta}\left(x - \frac{at^2}{2}\right) + at; \tag{10}$$

$$z = \frac{x - at^2/2}{\sqrt{4\chi t + \beta}}. \tag{11}$$

After this, at the front of the strong shock wave we have ([3])

$$D_y = \frac{dx}{dt} = \frac{k+1}{2}u = \frac{(k+1)x}{4\chi t + \beta}\left(x - \frac{at^2}{2}\right) + \frac{k+1}{2}at.$$

Integrating, we obtain the law of motion of the front of the strong shock wave

$$x = (4\chi t + \beta)^{(k+1)/4}\left[\mathrm{const} + \frac{k+1}{2}a\int\left(t - \frac{\chi t^2}{4\chi t + \beta}\right)(4\chi t + \beta)^{-(k+1)/4}\,dt\right].$$

Since at $t = 0$, $x = 0$, we finally obtain:

$$x(t) = \frac{a}{2(7-k)(3-k)}\left\{\frac{(k-1)\beta^2}{\chi^2}\left(\left[\frac{4\chi t + \beta}{\beta}\right]^{(k+1)/4} - 1\right) + (k+1)t\left[3t(3-k) - \frac{(k-1)\beta}{\chi}\right]\right\}. \tag{12}$$

Expanding the quantity $\left(1 + \frac{4\chi t}{\beta}\right)^{(k+1)/4}$ in a series and retaining only first-order terms of smallness in $\chi t/\beta$, which for $z \ll 1$ is sufficient for our solution, we find

$$x(t) = \frac{k+1}{4}at^2 + \frac{(k^2-1)\chi a t^3}{12\beta}. \tag{13}$$

This expression will give the law of motion of the shock front with sufficient accuracy; in this case the velocity of the wave front will be

$$D_y = \frac{k+1}{2}at + \frac{(k^2-1)}{4\beta}\chi a t^2, \tag{14}$$

and the velocity of the gas flow behind the wave front

$$u_{\mathrm{p}}(x,t) = at + \frac{(k-1)\chi a t^2}{2\beta}. \tag{15}$$

We now determine $\Phi(z)$, starting from the condition

$$\rho = \rho_{\mathrm{p}} = \frac{k+1}{k-1}\rho_0.$$

Then from (2)

$$\Phi(z) = \frac{k+1}{k-1} \frac{\rho_0}{\sqrt{4\chi t + \beta}},$$

where, taking (13) into account,

$$z = \frac{at^2(k-1)}{4\sqrt{4\chi t + \beta}} \left[1 + \frac{(k+1)}{3\beta}\chi t\right]. \tag{16}$$

From (16) we find $t = t_0(z)$. Thus,

$$\Phi(z) = \frac{k+1}{k-1} \frac{\rho_0}{\sqrt{4\chi t_0(z) + \beta}}.$$

Finally we obtain

$$\rho = \frac{k+1}{k-1} \sqrt{\frac{4\chi t + \beta}{4\chi t_0(z) + \beta}}\, \rho_0. \tag{17}$$

We proceed to determine $P_0(t)$. Since at the front of the shock wave [3]

$$P + \frac{H^2}{8\pi} = \frac{2\rho_0 D_y^2}{k+1} = \frac{2\rho_0}{k+1} \left[\frac{k+1}{2} at + \frac{k^2-1}{4} \frac{\chi a t^2}{\beta}\right]^2 = \frac{k+1}{2} \rho_0 a^2 t^2 \left[1 + \frac{k-1}{2} \frac{\chi t}{\beta}\right]^2, \tag{18}$$

then

$$P_0(t) = \frac{k+1}{2} \rho_0 a^2 t^2 \left(1 + \frac{k-1}{2} \frac{\chi t}{\beta}\right) + \int (u_t + u u_x) \rho \, dx.$$

The last integral must be calculated in general form and then the value of x at the front of the shock wave substituted. This gives:

$$P + \frac{H^2}{8\pi} = \frac{k+1}{2} \rho_0 a^2 t^2 \left(1 + \frac{k-1}{2} \frac{\chi t}{\beta}\right) +$$

$$+ \frac{k+1}{k-1} \rho_0 \int_x^{\tilde{x}} \sqrt{\frac{4\chi t + \beta}{4\chi t_0(z) + \beta}} \left(a - \frac{4\chi^2 (x - at^2/2)}{(4\chi t + \beta)^2}\right) dx, \tag{19}$$

where

$$\tilde{x} = \frac{k+1}{4}at^2 + \frac{k^2-1}{12}\frac{\chi a t^3}{\beta},$$

and the upper limit corresponds to the value $x = \tilde{x}$, which is chosen when calculating $P_0(t)$.

It remains to determine $A_0(t)$ in equation (1), whereby the function $H = H(x,t)$ will be completely determined. Since at the piston $P_\mathrm{p} = H^2/8\pi$, it follows from equation (19) that

$$\frac{H^2}{4\pi} = \eta_1(t), \tag{20}$$

where $\eta_1(t)$ is a known function of time t. On the other hand, substituting into (1) the value $x = at^2/2$, we obtain

$$H = A_0(t)\eta_2(t), \tag{21}$$

where $\eta_2(t)$ is also a known function of time t. Comparing (20) and (21), we find that

$$A_0(t) = \frac{2\sqrt{\pi\eta_1}}{\eta_2} = 2\sqrt{\pi\eta^*(t)}. \tag{22}$$

Thus, the stated problem of finding all the parameters determining the motion of a conducting plasma under the action of a piston has been solved.

Schmidt Institute of Physics of the Earth,
Academy of Sciences of the USSR

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CITED LITERATURE

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2. K. P. Stanyukovich, *Unsteady Motions of Continuous Media*, 1955, § 23.

3. F. A. Baum, S. A. Kaplan, K. P. Stanyukovich, *Introduction to Cosmic Gas Dynamics*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.